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Toeplitz and Hankel operators with symbols of measures

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Abstract

Toeplitz and Hankel operators with symbols of measures

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The main topics covered in this thesis are the definitions of Toeplitz and Hankel operators with symbols of complex Borel measures and their various properties. When a complex Borel measure μ on the unit circle is given, we define a Toepllitz operator T_μ, whose symbol is μ, as an unbounded linear operator on H^2. In that case, T_μ may not be densely defined. Nevertheless, we see that the domain of T_μ has always a special form. The central question in this thesis is to ask when the Toeplitz operator T_μ defined as a linear operator is bounded on its domain. The answer to this question is related to the compatibility of the symbol μ: T_μ is bounded if and only if μ is a Carleson measure on $\mathbb{T}$, when the domain of T_μ contains all polynomials. In addition, we investigate a connection between trigonometric moment problem and Toeplitz operators with symbols of complex Borel measures. We also provide corresponding definitions for Hankel operators, and then vertify various properties.

Keywords: Toeplitz operator, Hankel operator, Hardy space, Reproducing kernel, Unbounded operator, Complex Borel measure, Carleson measure, Moment problem
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Chapter 1
Introduction

A classical Toeplitz operator is the compression of a multiplication operator on the Lebesgue space $L^2(\mathbb{T})$ of the unit circle $\mathbb{T}$ to the Hardy space $H^2(\mathbb{T})$. The study of Toeplitz operators has been originated from the paper of O. Toeplitz [Toe10]. In the paper [Toe11], he used Toeplitz matrices to characterize the Fourier coefficients of nonnagative functions on the unit circle. Since then, much reseach on Toeplitz operators has been carried out by many authors. (cf. [Win42], [HW50], [HW54], [Ros62], [Wid64], [BD65], etc.). Studies of them have shown the fruitful relationship between Toeplitz operators and their symbols. The remarkable paper of A. Brown and P. R. Halmos [BH64] shows the algebraic properties of Toeplitz operators. Recently, the theory of Toeplitz operators has been studied in a variety of settings and connections with the other fields. One direction is to deals with Toeplitz operators on reproducing kernel spaces like Bergman spaces, Dirichlet spaces, or Fock spaces, etc. (cf. [ACM82], [RW92], [Cao99], [DL04], [Str92]). Another direction is to study Toeplitz operators with operator-valued symbols (cf. [CHL12], [CHDL14], [CHL14], [CHL16]). Also, truncated Toeplitz operators have attracted an attention. One of the most systematic papers on truncated Toeplitz operators is D. Sarason's paper in 2007 ([Sar07]). In that paper, he has used "compatible" measures to describe bounded truncated Toeplitz operators. The boundedness of infinite Hankel matrices is also related to the compatibility of measures: The infinite Hankel matrix of the moment of a nonnegative Carleson measure is bounded and vice versa [Wid66]. These facts invites us to consider Toeplitz and Hankel operators with symbols of measures.

The main purpose of this paper is to explore properties of Toeplitz and Hankel operators with symbols of measures. In this study, unbounded linear operators arise naturally. When studying unbounded Toeplitz operators, it was usually considered that the symbols come from $L^2(\mathbb{T})$. In 2008, D. Sarason [Sar08] has treated not only the case of $L^2(\mathbb{T})$ symbols but the case of analytic functions on $\mathbb{D}$. It is natural to attempt to extend the symbols of Toeplitz and Hankel operators to measures, because the initial reasearch for them was related to the moment problem. As mentioned before, Toeplitz and Hankel operators associated with measures can be seen in the papers [Wid66] and [Sar07]. In this thesis we provide an explicit definition of Toeplitz and Hankel operators with symbols of measures, and then consider the unbounded operator theory of them.

Our first consideration for the symbol of a Toeplitz operator, denoted by T_μ, is a complex Borel measure μ on the unit circle. With this symbol, a Toeplitz operator becomes a linear operator, from a linear subspace of H^2 into H^2, which is possibly unbounded. When we study an unbounded linear operator, we usually assume that its domain is dense, i.e., the operator is densely defined. Hence one may ask if T_μ is densely defined. Toeplitz operators with L^2 symbols are always densely defined. Unlike when the symbol is a function, it is not easy to answer the question. Nonetheless, we will show that the domain is represented in the form of one of three cases (cf. Proposition 3.8). In particular, we can show that if the domain of T_μ contains at least one polynomial, then T_μ is densely defined (cf. Proposition 3.9).

We can then ask when T_μ is the bounded. From the results of [BH64], we knew that a Toeplitz operator with L^2-symbols is bounded if and only if the symbol is essentially bounded. About the question on the boundedness of T_μ, we may get an idea from Widom's result for Hankel operators, which tells us that the boundedness of an infinite Hankel matrix is related with the "compatibility" of the moment measure μ. Indeed, we can obtain the corresponding result for T_μ: T_μ is bounded if and only if μ is a Carleson measure on $\mathbb{T}$ and the domain of T_μ contains all polynomials (cf. Theorem 3.28). Consequently, T_μ is a restriction of a Toeplitz operator with L^∞-symbol.

If we omit the condition that the domain contains all polynomials, the situation is different. For example, if μ is the unit mass concentrated at $z = 1$, then the Toeplitz operator T_μ is densely defined and trivial, i.e., the range is zero (cf. Example 3.4(b)). Hence T_μ is bounded on its domain, but μ is not a Carleson measure. In fact, we show most singular measures induce a trivial linear operator (cf. Proposition 3.15).

The moment problem is also related to Toeplitz and Hankel operators. The moment problem is to find a measure whose moment is given sequence, or is to find a condition on measure to be a moment measure. The trigonometric moment problem and Hamburger moment problem, which are well known moment problems . Their solution are related to the nonnegativeness of certain forms of matrices. More precisely, if (α_j) is a sequence of complex numbers, then the condition that the sequence is representable by a nonnegative measure on $\mathbb{R}$ is that the Hankel matrix (α_{j+k}) is nonnegative definite. If the matrix is bounded, then the moment measure may be found as a Carleson measure (cf. [Pel], [Wid66]). We show the corresponding description for trigonometric moment problem (cf. Proposition 3.31).

We also define truncated Toeplitz operators using measures on $\mathbb{T}$, and then verifty that a Carleson measure for the domain induces bounded truncated Toeplitz operator when the domain is finite dimensional (cf. Section 3.5).

If we extend the class of symbols for Toeplitz operators to complex Borel measures on the closed unit disc, the corresponding matrix for a Toeplitz operator may not be an infinite Toeplitz matrix. Nevertheless, we show that the boundedness of a Toeplitz operator $\mathcal{T}_\mu$ is related to the "compatibility" of its symbol μ (cf. Theorem 4.5).

The organization of this paper is as follows:

Chapter 2 is the preliminary chapter. In that chapter, we recall some notations, notions, definitions, and various known facts. Lebesgue spaces, Hardy spaces, measures, unboudned operators, Toeplitz and Hankel matrices, etc. is contained.

In Chapter 3 we define Toeplitz operators with symbols of complex Borel measures on $\mathbb{T}$. We then investigate various properties of Toeplitz operators, especially for their domains, spectral properties, and boundedness.

In Chapter 4 we consider complex Borel measures on the closed unit disc. We then define Hankel operators with symbols of complex Borel measures on $\overline{\mathbb{D}}$. Much discussion for Hankel operators is similar to Chapter 3.

In Chapter 5, we give concluding remaks and unsolved problems.

Chapter 2
Preliminaries

In this chapter, we introduce some basic notions and definitions. We also take a look at several known facts to use later. Lebesgue spaces, Hardy spaces, complex Borel measures, unboudned operators, Toeplitz and Hankel operators, etc. is contained.

2.1 Lebesgue spaces and Hardy spaces

Let $\mathbb{N}_0$ denote the set of all nonnegative integers, i.e., $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. For $0 < p \leq \infty$, we write

$$\ell^p \equiv \ell^p(\mathbb{N}_0).$$

That is, ℓ^p is the set of all sequences $x = (x_n)_{n\in\mathbb{N}_0}$ of complex numbers for which $\|x\|_{\ell^p} < \infty$, where

$$\|x\|_{\ell^p} = \begin{cases} \left(\sum_{n=0}^{\infty} |x_n|^p \right)^{\frac{1}{p}} & \text{if} \quad 0 < p < \infty, \\ \sup_{n\in\mathbb{N}_0} |x_n| & \text{if} \quad p = \infty. \end{cases}$$

Let $\mathbb{T}$ be the unit circle in the complex plane $\mathbb{C}$. Let m be the normalized Lebesgue measure on $\mathbb{T}$, so that $m(\mathbb{T}) = 1$. For $0 < p \leq \infty$, we write

$$L^p(\mathbb{T}) \equiv L^p(\mathbb{T}, m)$$

for the Lebesgue space on $\mathbb{T}$. That is, $L^p(\mathbb{T})$ is the set of all complex m-measurable functions f on $\mathbb{T}$ for which $\|f\|_p < \infty$, where

$$\|f\|_p = \begin{cases} \left(\int_{\mathbb{T}} |f|^p \, dm \right)^{\frac{1}{p}} & \text{if} \quad 0 < p < \infty, \\ \operatorname*{ess\,sup}_{\zeta \in \mathbb{T}} |f(\zeta)| & \text{if} \quad p = \infty. \end{cases}$$

Note that the set of all trigonometric polynomials is dense in $L^p(\mathbb{T})$ whenever $0 < p < \infty$. On the other hand, the closure of the set of all trigonometric polynomials in $L^\infty(\mathbb{T})$ is the set $C(\mathbb{T})$ of all continuous functions on $\mathbb{T}$.

For $f \in L^1(\mathbb{T})$ and $n \in \mathbb{Z}$, let $\widehat{f}(n)$ denote the n-th Fourier coefficient of f, i.e.,

$$\widehat{f}(n) = \int_{\mathbb{T}} f(\zeta)\overline{\zeta^n} \, dm(\zeta).$$

If $1 \le p \le \infty$, the spaces

$$H^p(\mathbb{T}) = \left\{ f \in L^p(\mathbb{T}) : \widehat{f}(n) = 0,\, n < 0 \right\}$$

and

$$H_0^p(\mathbb{T}) = \left\{ f \in L^p(\mathbb{T}) : \widehat{f}(n) = 0,\, n \ge 0 \right\}$$

are closed subspaces of $L^p(\mathbb{T})$. For any space X of complex-valued functions in this thesis, we write

$$\overline{X} = \left\{ \overline{f} : f \in X \right\},$$

where $\overline{f}$ is the complex conjugation of f. Thus

$$\overline{H^p(\mathbb{T})} = \left\{ f \in L^p(\mathbb{T}) : \widehat{f}(n) = 0,\, n > 0 \right\}$$

and

$$\overline{H_0^p(\mathbb{T})} = \left\{ f \in L^p(\mathbb{T}) : \widehat{f}(n) = 0,\, n \ge 0 \right\},$$

which are also closed subspaces of $L^p(\mathbb{T})$.

Let $\mathbb{D}$ be the open unit disc and $\overline{\mathbb{D}}$ is the closed unit disc in the complex plane. Let $H(\mathbb{D})$ denote the class of all analytic functions on $\mathbb{D}$ with the topology of locally uniform convergence; for a sequence (f_j) in $H(\mathbb{D})$, we say that $f_j \to f$ in $H(\mathbb{D})$ when (f_j) converges uniformly to f on each compact subset of $\mathbb{D}$. Let $A(\mathbb{D})$ denote the disc algebra, i.e., the set of all continuous functions on $\overline{\mathbb{D}}$ which is analytic on $\mathbb{D}$. Note that $A(\mathbb{D})$ is identified with the set

$$\{f \in C(\mathbb{T}) : \widehat{f}(n) = 0,\, n < 0\}.$$

Note also that $A(\mathbb{D})$ is the closure of the set of all polynomials with respect to the supremum norm (cf. [Con1], [Hof]).

For $0 < p \leq \infty$, we write

$$H^p \equiv H^p(\mathbb{D})$$

for the Hardy space on $\mathbb{D}$. This is, H^p is the set of all analytic functions f on $\mathbb{D}$ for which

$$\|f\|_{H^p} = \sup_{0<r<1} \|f_r\|_p,$$

where $f_r(\zeta) = f(r\zeta)$ on $\mathbb{T}$. Note that $A(\mathbb{D}) \subset H^\infty \subset H^p \subset H^s$ when $0 < s < p < \infty$. If $1 \leq p \leq \infty$, then H^p is a Banach space with the norm $\|\cdot\|_{H^p}$. Note that the set of all polynomials is dense in H^p (cf. [Dur]). Since the disc algebra $A(\mathbb{D})$ contains every polynomial, it is dense in H^p for each $1 \leq p < \infty$. On the other hand, $A(\mathbb{D})$ is a closed subalgebra of the Banach algebra H^∞.

A crucial feature of the Hardy spaces is the identification of H^p-functions with their nontangential limit functions (cf. [Dur], [Rud]): If $f \in H^p$, then it has the nontangential limit $f^*(\zeta)$ at almost every point $\zeta \in \mathbb{T}$ (with respect to m), the nontangential limit function f^* belongs to $L^p(\mathbb{T})$, and $\|f^*\|_p = \|f\|_{H^p}$.

In the case $p \geq 1$, we can say more about the Hardy space H^p. If $f \in H^p$ and the power series expansion of f is given by

$$f(z) = \sum_{n=0}^{\infty} c_n z^n \qquad (z \in \mathbb{D}),$$

then $f^* \in H^p(\mathbb{T})$ and $\widehat{f^*}(n) = c_n$ for all $n \in \mathbb{N}_0$. In particular, we can restore the function f by using the Cauchy integral formula and the Poisson integral formula: If $f \in H^1$, then

$$f(z) = \int_{\mathbb{T}} \frac{f^*(\zeta)}{1 - \bar{\zeta} z}\, dm(\zeta) \qquad (z \in \mathbb{D})$$

and

$$f(z) = \int_{\mathbb{T}} \mathrm{Re}\left[\frac{\zeta + z}{\zeta - z}\right] f^*(\zeta)\, dm(\zeta) \qquad (z \in \mathbb{D}).$$

In fact, the mapping $f \mapsto f^*$ is an isometric isomorphism between two Banach spaces H^p and $H^p(\mathbb{T})$. Hence, we identify functions f in H^p with their nontangential limit functions f^* in $H^p(\mathbb{T})$, and often write f instead of f^*. Also, we write $\|f\|_p$ for the H^p-norm $\|f\|_{H^p}$.

Another important feature of the Hardy spaces H^p is the inner-outer factorization. A function $\theta \in H^\infty$ is called an *inner* function if $|\theta^*| = 1$ a.e. on $\mathbb{T}$. For convenience we regard the zero function 0 as an inner function. If $k \in \mathbb{N}_0$ and if $\{\alpha_j\}_{j \in \mathbb{N}}$ is a sequence in $\mathbb{D} \setminus \{0\}$ such that $\sum_{j=1}^{\infty}(1 - |\alpha_j|) < \infty$, the function

$$B(z) = z^k \prod_{j=1}^{\infty} \frac{\alpha_j - z}{1 - \bar{\alpha}_j z} \frac{|\alpha_j|}{\alpha_j} \qquad (z \in \mathbb{D}),$$

which is called a *Blaschke product*, is an inner function. A inner function is said to be *singular* if it has no zero in $\mathbb{D}$. Every singular inner function is of the form

$$S(z) = c \cdot \exp\left\{ - \int_{\mathbb{T}} \frac{\zeta + z}{\zeta - z}\, d\mu(\zeta) \right\} \qquad (z \in \mathbb{D}),$$

where c is a unimodular constant and μ is a finite nonnegative Borel measure on $\mathbb{T}$ which is singular. Every nonzero inner function θ is of the form $\theta = BS$, where B is a Blaschke product and S is a singular inner function. A function Φ is called an *outer* function if it takes the form

$$\Phi(z) = c \cdot \exp\left\{ \int_{\mathbb{T}} \frac{\zeta + z}{\zeta - z} \log \varphi(\zeta)\, dm(\zeta) \right\} \qquad (z \in \mathbb{D}),$$

where c is a unimodular constant and φ is a positive m-measurable function on $\mathbb{T}$ such that $\log \varphi \in L^1(\mathbb{T})$. Note that $\Phi \in H^p$ if and only if $\varphi \in L^p(\mathbb{T})$, in which case $\|\Phi\|_p = \|\varphi\|_p$. For $0 < p \le \infty$, every function $f \in H^p$ can be expressed in the form

$$f = \theta\Phi,$$

where θ is an inner function and Φ is an outer function; this factorization is unique up to a unimodular constant. (Cf. [Rud].)

2.2 The Hardy-Hilbert space H^2

The Lebesgue space $L^2(\mathbb{T})$ is a Hilbert space with the inner product

$$\langle f, g \rangle_{L^2(\mathbb{T})} = \int_{\mathbb{T}} f \overline{g}\, dm.$$

The set $\{z^n : n \in \mathbb{Z}\}$ is the standard orthonormal basis of $L^2(\mathbb{T})$. (We often use the symbol z to denote the identity function on a subset of the complex plane.) Parseval's theorem tells us that the mapping $f \mapsto \widehat{f}$ is a Hilbert space isomorphism between $L^2(\mathbb{T})$ and $\ell^2(\mathbb{Z})$.

Since $H^2(\mathbb{T})$ is a closed subspace of $L^2(\mathbb{T})$, it follows that the Hardy space H^2 is also a Hilbert space with the inner product

$$\langle f, g \rangle_{H^2} = \langle f^*, g^* \rangle_{L^2(\mathbb{T})} = \int_{\mathbb{T}} f^* \overline{g^*}\, dm \qquad (f, g \in H^2).$$

The set $\{1, z, z^2, \dots\}$ is the standard orthonormal basis of H^2. It follows from the Parseval's theorem, the mapping $f \mapsto \widehat{f}$ is a Hilbert space isomorphism between H^2 and ℓ^2. In particular, if $f, g \in H^2$, and $f = \sum_{n=0}^{\infty} a_n z^n$, $g = \sum_{n=0}^{\infty} b_n z^n$, then

$$\langle f, g \rangle_{H^2} = \sum_{n=0}^{\infty} a_n \overline{b_n}.$$

For each λ, the function k_λ, given by

$$k_\lambda(z) = \frac{1}{1 - \overline{\lambda} z} \qquad (z \in \mathbb{D}),$$

is the reproducing kernel for H^2 at λ (cf. [AM]). Hence if $f \in H^2$, then

$$\langle f, k_\lambda \rangle_{H^2} = f(\lambda).$$

If $f \in H^2$ and $f \perp k_\lambda$ for all $\lambda \in \mathbb{D}$, then $f(\lambda) = \langle f, k_\lambda \rangle_{H^2} = 0$ for all $\lambda \in \mathbb{D}$, and hence $f = 0$. It follows that the linear span of the set $\{k_\lambda : \lambda \in \mathbb{D}\}$ is dense in H^2.

In the Hardy space H^2, there are at least three kinds of convergence. Suppose that (f_j) is a sequence in H^2 and $f \in H^2$.

- $f_j \to f$ *strongly* if and only if $f_j \to f$ in H^2, i.e., $\|f - f_j\|_2 \to 0$.
- $f_j \to f$ *weakly* if and only if $\langle f_j, g \rangle_{H^2} \to \langle f, g \rangle_{H^2}$ for each $g \in H^2$.
- $f_j \to f$ in $H(\mathbb{D})$ if and only if $f_j \to f$ uniformly on each compact subset of $\mathbb{D}$.

Of course, the third convergence implies the pointwise convergence.

If $f_j \to f$ strongly, i.e., $\|f - f_j\|_2 \to 0$, then, for any $g \in H^2$,

$$\left|\langle f, g \rangle_{H^2} - \langle f_j, g \rangle_{H^2}\right| = \left|\langle f - f_j, g \rangle_{H^2}\right| \leq \|f - f_j\|_2 \|g\|_2,$$

and hence $\langle f_j, g \rangle_{H^2} \to \langle f, g \rangle_{H^2}$, i.e., $f_j \to f$ weakly. The converse may fail in general. For a counterexample, consider the functions

$$f_j = z^j \qquad (j \in \mathbb{N}).$$

If $g \in H^2$, then $\widehat{g} \in \ell^2$, in particular, $\widehat{g}(j) \to 0$. Thus $\langle f_j, g\rangle_{H^2} = \overline{\langle g, z^j\rangle}_{H^2} = \overline{\widehat{g}(j)} \to 0$. It follows that $f_j \to 0$ weakly. But $\|f_j\|_2 = 1$ for all j. Hence $\|f_j - 0\|_2 \not\to 0$.

Suppose that $f_j \to f$ weakly. Put $g_j = f - f_j$. Then $g_j \to 0$ weakly. We show that $g_j \to 0$ in $H(\mathbb{D})$. Since every compact subset of $\mathbb{D}$ is contained in some closed disc $\overline{D}(0;r)$, where $r < 1$, it suffices to show that $g_j \to 0$ uniformly on $\overline{D}(0;r)$ for each $0 < r < 1$. By the principle of uniform boundedness (cf. [Hal], [Rud]), there exists a constant $M > 0$ such that $\|g_j\|_2 \le M$. Also, we have $g_j(z) = \langle g_j, k_z\rangle_{H^2} \to 0$ for each $z \in \mathbb{D}$. Now fix $0 < r < s < 1$. If $|z| \le s$, then

$$|g_j(z)| = \left|\langle g_j, k_z\rangle_{H^2}\right| \le \|g_j\|_2 \|k_z\|_2 \le \frac{M}{\sqrt{1-s^2}}$$

for every j. Put $h_j(z) = g_j(sz)$, then $h_j \in A(\mathbb{D})$. If $|z| \le r$, it follows that

$$|g_j(z)| = |h_j(z/s)| = \left| \int_{\mathbb{T}} \frac{h_j(\zeta)}{1 - \bar{\zeta} z/s}\, dm(\zeta)\right| \le \int_{\mathbb{T}} \frac{|g_j(s\zeta)|}{1 - r/s}\, dm(\zeta).$$

By the dominated convergence theorem, the last integral tends to 0. Therefore $g_j \to 0$ uniformly on the closed disc $\overline{D}(0;r)$. Since r was arbitrary, it follows that $g_j \to 0$ in $H(\mathbb{D})$, i.e., $f_j \to f$ in $H(\mathbb{D})$. We conclude that if $f_j \to f$ weakly, then $f_j \to f$ in $H(\mathbb{D})$.

The converse may fail in general. For a counterexample, consider the functions

$$f_j = j z^j \qquad (j \in \mathbb{N}).$$

Let $0 < r < 1$. Sicne $|f_j(z)| \le j r^j$ for all $z \in \overline{D}(0;r)$ and $j r^j \to 0$, it follows that $f_j \to 0$ uniformly on $\overline{D}(0;r)$. Therefore $f_j \to 0$ in $H(\mathbb{D})$. On the other hand, consider the function $g = \sum_{n=1}^{\infty} \frac{1}{n} z^n$. Since $\sum_{n=1}^{\infty} \frac{1}{n^2} < \infty$, we have $g \in H^2$. Observe that

$$\langle f_j, g\rangle_{H^2} = \overline{\langle g, j z^j\rangle}_{H^2} = 1$$

for all $j \ge 1$. Hence $\langle f_j, g\rangle_{H^2} \not\to 0$. This shows that $f_j \not\to 0$ weakly.

2.3 Shift operators

The shift operator and its adjoint are one of the most interesting operators on the Hardy space. We refer the reader to the text [Nik] which treats the shift operator in great detail. For convenience, we define them on $H(\mathbb{D})$. For $f \in H(\mathbb{D})$, define

$$Sf(z) := zf(z) \qquad (z \in \mathbb{D}),$$

and

$$S^*f(z) := \frac{f(z) - f(0)}{z} \qquad (z \in \mathbb{D}).$$

The operators S and S^* are often called the *unilateral shift* and *backward shift*, respectively. One can easily verify the following properties of S and S^*:

- S and S^* are linear transformations on the vector space $H(\mathbb{D})$.
- S is injective and S^* is surjective.
- $S^*Sf = f$ and $SS^*f = f - f(0)$ for every $f \in H(\mathbb{D})$.
- The Hardy spaces H^p $(0 < p \leq \infty)$ and the disc algebra $A(\mathbb{D})$ are invariant for both S and S^*.
- S is an isometry on the Banach spaces H^p $(1 \leq p \leq \infty)$.
- S^* is a contraction on the Hilbert space H^2.
- As bounded operators on H^2, S and S^* are the adjoint operators of each other.

For the unilateral shift S on H^2, the spectrum is $\sigma(S) = \overline{\mathbb{D}}$, the point spectrum is $\sigma_{\mathrm{p}}(S) = \varnothing$, and the approximate point spectrum is $\sigma_{\mathrm{ap}}(S) = \mathbb{T}$; for the backward shift S^* on H^2, $\sigma(S^*) = \overline{\mathbb{D}}$, $\sigma_{\mathrm{p}}(S^*) = \mathbb{D}$, and $\sigma_{\mathrm{ap}}(S) = \overline{\mathbb{D}}$. (cf. [Hal]).

One of the most remarkable theorem in analysis is the Beurling's theorem (cf. [Beu49], [Hel], [Rud]), which characterizes all S-invariant subspaces of H^2. (We use the term "subspace" for a closed linear subspace.) For a subspace M of H^2, M is

S-invariant if and only if $M = \theta H^2$ for some inner function $\theta \in H^\infty$. It follows that the orthogonal complements

$$\mathcal{H}(\theta) := H^2 \ominus \theta H^2$$

constitute the family of all S^*-invariant subspaces. One nontrivial fact about $\mathcal{H}(\theta)$ is the Aleksandrov's density theorem which says that, for every inner function θ, the space

$$\mathcal{H}_A(\theta) := \mathcal{H}(\theta) \cap A(\mathbb{D})$$

is dense in $\mathcal{H}(\theta)$ (cf. [Alek95], [CMR, Section 8.5]).

As a consequence of Beurling's theorem, we obtain the following characterization of outer H^2-functions: For a function $f \in H^2$, f is an outer function if and only if the linear span of the set $\{S^n f : n = 0, 1, 2, \dots\}$ is dense in H^2.

2.4 Complex Borel measures

Let X be a subset of the complex plane. We denote by $\mathscr{B}_X$ the σ-algebra of all Borel sets in X. Recall that a complex Borel measure on X is a complex-valued countably additive function on the σ-algebra $\mathscr{B}_X$. Most of measures appearing in this thesis are supported on either the unit circle $\mathbb{T}$ or the unit disc $\overline{\mathbb{D}}$. In this section, we review several facts about complex Borel measures on $\mathbb{T}$, Cauchy transforms of measures, and Carleson measures.

Let $M(\mathbb{T})$ be the set of all complex Borel measures on $\mathbb{T}$. Note that $M(\mathbb{T})$ is a Banach space with the total variation norm $\|\mu\| = |\mu|(\mathbb{T})$, where $|\mu|$ is the total variation measure of μ. The Riesz representation theorem tells us that $M(\mathbb{T})$ is isomorphic to the dual space of the Banach space $C(\mathbb{T})$ of continuous functions of $\mathbb{T}$ equipped with the supremum norm. As a dual space, $M(\mathbb{T})$ has the weak-$*$ topology; we say that a sequence (μ_j) in $M(\mathbb{T})$ converges weak-$*$ to μ when

$$\int_{\mathbb{T}} f\,d\mu_j \to \int_{\mathbb{T}} f\,d\mu$$

for each $f \in C(\mathbb{T})$.

If $\mu \in M(\mathbb{T})$ is a nonnegative measure, the support of μ is the closed set

$$\operatorname{supp}\mu = \big\{\zeta \in \mathbb{T} : \mu(I) > 0 \text{ for all open arcs } I \subseteq \mathbb{T} \text{ whose centers are at } \zeta\big\};$$

if $\mu \in M(\mathbb{T})$ is a complex measure, $\operatorname{supp}\mu = \operatorname{supp}|\mu|$.

Let $\mu \in M(\mathbb{T})$. For any function $f \in L^1(\mathbb{T}, |\mu|)$, let $f \cdot \mu$ denote the complex Borel measure on $\mathbb{T}$ defined by

$$(f \cdot \mu)(E) = \int_E f\,d\mu \qquad (E \in \mathscr{B}_{\mathbb{T}}).$$

Then $|f \cdot \mu| = |f| \cdot |\mu|$. Note that for every $f \in C(\mathbb{T})$, the measure $f \cdot \mu$ is defined and $\|f \cdot \mu\| \leq \|f\|_\infty \|\mu\|$.

We may regard the normalized Lebesgue measure m as a finite nonnegative Borel measure. Hence $m \in M(\mathbb{T})$. Recall that a measure $\mu \in M(\mathbb{T})$ is said to be *absolutely continuous* (with respect to m), and write

$$\mu \ll m$$

if $\mu(E) = 0$ for every $E \in \mathscr{B}_{\mathbb{T}}$ for which $m(E) = 0$; a measure $\mu \in M(\mathbb{T})$ is said to be *singular* (with respect to m), and write

$$\mu \perp m$$

if μ is concentrated on a set of (Lebesgue) measure zero. Every measure $\mu \in M(\mathbb{T})$ has the Lebesgue decomposition

$$\mu = \mu_{\mathrm{a}} + \mu_{\mathrm{s}},$$

where $\mu_a \ll m$ and $\mu_s \perp m$. The Radon-Nikodym theorem shows that every absolutely continuous measure μ is represented by $\mu = f \cdot m$, where $f \in L^1(\mathbb{T})$. In this case, f is called the Radon-Nikodym derivative of μ (with respect to m), and written by

$$f = \frac{d\mu}{dm}.$$

We often identify the absolutely continuous measure $\mu \in M(\mathbb{T})$ with its derivative $\frac{d\mu}{dm} \in L^1(\mathbb{T})$.

A measure $\mu \in M(\mathbb{T})$ is said to be *discrete* if it is concentrated on a countable subset of $\mathbb{T}$. Since every countable set has (Lebesgue) measure zero, every discrete measure is singular. For any point $\zeta \in \mathbb{T}$, we denote the unit mass concentrated at ζ by δ_ζ, i.e,

$$\delta_\zeta(E) = \begin{cases} 1 & \text{if} \quad \zeta \in E, \\ 0 & \text{if} \quad \zeta \notin E. \end{cases}$$

for every $E \in \mathscr{B}_{\mathbb{T}}$. Clearly, δ_ζ is discrete. Moreover, every discrete measure $\mu \in M(\mathbb{T})$ is of the form

$$\mu = \sum_j c_j \delta_{\zeta_j},$$

where c_j are nonzero complex numbers such that $\sum_j |c_j| < \infty$ and ζ_j are distinct points of $\mathbb{T}$.

A measure $\mu \in M(\mathbb{T})$ is said to be *continuous* if $\mu(\{\zeta\}) = 0$ for all $\zeta \in \mathbb{T}$. It is clear that every absolutely continuous measure is continuous. However, there are singular continuous measures. (See, for example, [BM74].) Any singular measure is the sum of continuous measure and discrete measure. It follows that every $\mu \in M(\mathbb{T})$ can be written as the sum of measures

$$\mu = \mu_{\mathrm{ac}} + \mu_{\mathrm{sc}} + \mu_{\mathrm{d}},$$

where $\mu_{\rm ac}$ is absolutely continuous, $\mu_{\rm sc}$ is singular continuous, and $\mu_{\rm d}$ is discrete.

For $\mu \in M(\mathbb{T})$, the nth Fourier coefficient of μ is given by

$$\widehat{\mu}(n) = \int_{\mathbb{T}} \bar{\zeta}^n \, d\mu(\zeta) \qquad (n \in \mathbb{Z}).$$

For any $\mu \in M(\mathbb{T})$ the sequence $\widehat{\mu} = (\widehat{\mu}(n))_{n\in\mathbb{Z}}$ is bounded and the mapping $\mu \mapsto \widehat{\mu}$ is a bounded linear transformation from $M(\mathbb{T})$ into $\ell^\infty(\mathbb{Z})$. Note that the mapping $\mu \mapsto \widehat{\mu}$ is one-to-one, and hence a measure $\mu \in M(\mathbb{T})$ is completely determined by its Fourier coefficients. Note also that for a sequence (μ_j) in $M(\mathbb{T})$, $\mu_j \to \mu$ weak-$*$ if and only if $\widehat{\mu}_j(n) \to \widehat{\mu}(n)$ for each $n \in \mathbb{Z}$. By the theorem of F. and M. Riesz (cf. [Rud]), if $\mu \in M(\mathbb{T})$ is analytic, i.e., $\widehat{\mu}(n) = 0$ for all $n \le 0$, then $\mu \ll m$ and $\frac{d\mu}{dm} \in H^1(\mathbb{T})$, in other words, $\mu = f \cdot m$ for some $f \in H^1(\mathbb{T})$.

In the process of extending definitions of Toeplitz operators, we use the Cauchy transform as the "projection" of measures. We refer the reader to the text [CMR] for thorough treatments of the Cauchy transform. For $\mu \in M(\mathbb{T})$, the analytic function $P\mu$ on $\mathbb{D}$, given by

$$(P\mu)(z) = \int_{\mathbb{T}} \frac{1}{1-\bar{\zeta}z} \, d\mu(\zeta) = \sum_{n=0}^{\infty} \widehat{\mu}(n) z^n \qquad (z \in \mathbb{D}),$$

is called the *Cauchy transform* of μ. Clearly, the mapping P is a linear transformation from $M(\mathbb{T})$ into $H(\mathbb{D})$. Note that P is continuous relative to the weak-$*$ topology of $M(\mathbb{T})$ and the topology of locally uniform convergence of $H(\mathbb{D})$.

We may regard $f \in L^1(\mathbb{T})$ as the absolutely continuous measure $f \cdot m \in M(\mathbb{T})$. Hence we denote $P(f \cdot m)$ by Pf, i.e.,

$$(Pf)(z) = \int_{\mathbb{T}} \frac{f(\zeta)}{1-\bar{\zeta}z} \, dm(\zeta) = \sum_{n=0}^{\infty} \widehat{f}(n) z^n \qquad (z \in \mathbb{D}).$$

(Clearly, $\widehat{f \cdot m}(n) = \widehat{f}(n)$.) By the Cauchy integral formula, for every $f \in H^1(\mathbb{T})$,

$$(Pf)^* = f.$$

It is known that the mapping $P : L^p(\mathbb{T}) \to H^p$ is bounded whenever $1 < p < \infty$ (cf. [CR], [Hof]). It follows that

$$L^p(\mathbb{T}) = H^p(\mathbb{T}) \oplus \overline{H^p_0(\mathbb{T})}.$$

However, this equality is no longer valid if $p = 1$:

$$H^1(\mathbb{T}) \oplus \overline{H^1_0(\mathbb{T})} \subsetneq L^1(\mathbb{T}).$$

This follows from the fact that there exists a function $f \in L^1(\mathbb{T})$ such that $Pf \notin H^1$ (cf. [Hof]). Of course, $H^1(\mathbb{T}) \oplus \overline{H^1_0(\mathbb{T})}$ is dense in $L^1(\mathbb{T})$, because it contains all trigonometric polynomials. The special case where $p = 2$ is of particular importance. As we have identified H^2 with $H^2(\mathbb{T})$, the mapping P may be regarded as the orthogonal projection of $L^2(\mathbb{T})$ onto $H^2(\mathbb{T})$ (so called the *Riesz projection*).

A complex Borel measure on $\mathbb{D}$ is called a *Carleson measure* if there exists a constant $c > 0$ such that

$$|\mu|(S_{\theta_0,h}) \leq c \cdot h$$

for every sector

$$S_{\theta_0,h} = \left\{ re^{i\theta} : 1 - h \leq r < 1,\ |\theta_0 - \theta| \leq h \right\}.$$

The Carleson imbedding theorem (cf. [Car62], [Gar]) shows that a complex Borel measure μ on $\mathbb{D}$ is a Carleson measure if and only if there exists a constant $c > 0$ such that

$$\int_{\mathbb{T}} |f|^2 \, d|\mu| \leq c \cdot \|f\|_2^2$$

for every $f \in H^2$, or equivalently, the identical imbedding operator I_μ from H^2 into $L^2(\mathbb{D}, |\mu|)$, given by

$$I_\mu f = f \qquad (f \in H^2),$$

is bounded. If the identical imbedding operator I_μ is compact, the measure μ is called a *vanishing* Carleson measure.

In terms of the properties of the identical imbedding operator, we may define "a kind of" Carleson measures on a set other than $\mathbb{D}$. In [Sar07], Sarason has introduced that type of measures. Fix an inner function $\theta \in H^\infty$. A complex Borel measure μ on $\mathbb{T}$ is called a Carleson measure for $\mathcal{H}(\theta)$ if there exists a constant $c > 0$ such that

$$\int_{\mathbb{T}} |f|^2 \, d|\mu| \leq c \cdot \|f\|_2^2$$

for every $f \in \mathcal{H}_A(\theta)$. (Recall that $\mathcal{H}_A(\theta)$ is dense in $\mathcal{H}(\theta)$.) In other words, μ is a Carleson measure for $\mathcal{H}(\theta)$ if and only if the identical imbedding operator I_μ from $\mathcal{H}_A(\theta) \subseteq \mathcal{H}(\theta)$ into $L^2(\mathbb{T}, |\mu|)$, given by

$$I_\mu f = f \qquad (f \in \mathcal{H}_A(\theta)),$$

is bounded. If μ is a Carleson measure for $H^2 = \mathcal{H}(0)$, we say simply that μ is a $\mathbb{T}$-Carleson measure.

2.5 Unbounded operators

We briefly review various notions on theory of unbounded opeartors. We refer the reader to the text [Sch1] for details.

Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be Hilbert spaces, and let $\mathcal{D}(T)$ be a linear subspace of $\mathcal{H}_1$. Then a linear mapping $T : \mathcal{D}(T) \to \mathcal{H}_2$ is called a *linear operator from $\mathcal{H}_1$ into $\mathcal{H}_2$ with domain $\mathcal{D}(T)$*. The *kernel* of T is the

$$\ker T = \big\{x \in \mathcal{D}(T) : Tx = 0\big\},$$

the *range* of T is the

$$\operatorname{ran} T = \big\{Tx \in \mathcal{H}_2 : x \in \mathcal{D}(T)\big\},$$

and the *graph* of T is the set

$$\mathcal{G}(T) = \big\{(x, Tx) \in \mathcal{H}_1 \oplus \mathcal{H}_2 : x \in \mathcal{D}(T)\big\}.$$

The operator T is said to be *trivial* if $\operatorname{ran} T = \{0\}$, or equivalently, $\mathcal{D}(T) = \ker T$.

Let T_1 and T_2 be two linear operator from $\mathcal{H}_1$ into $\mathcal{H}_2$. We say that T_2 is an *extension* of T_1 and write $T_1 \subseteq T_2$, when $\mathcal{D}(T_1) \subseteq \mathcal{D}(T_2)$ and $T_1 x = T_2 x$ for all $x \in \mathcal{D}(T_1)$. By definition $T_1 = T_2$ if and only if $T_1 \subseteq T_2$ and $T_2 \subseteq T_1$.

Let T be a linear operator from $\mathcal{H}_1$ into $\mathcal{H}_2$ with domain $\mathcal{D}(T)$. Then T is said to be *closed* if its graph $\mathcal{G}(T)$ is a closed subset of the Hilbert space $\mathcal{H}_1 \oplus \mathcal{H}_2$, and T is said to be *closable* if there exists a closed extension. For a closable linear operator T, the smallest closed extension of T is called the *closure* of T and denoted by $\operatorname{cl} T$. Note that

$$\operatorname{cl}_{\mathcal{H}_1 \oplus \mathcal{H}_2}(\mathcal{G}(T)) = \mathcal{G}(\operatorname{cl} T).$$

The linear operator T is said to be *bounded* if there exists a constant $c > 0$ such that

$$\|Tx\|_{\mathcal{H}_2} \leq c \cdot \|x\|_{\mathcal{H}_1}$$

for every $x \in \mathcal{D}(T)$. Note that if T is bounded (on $\mathcal{D}(T)$), then it is closable and its closure $\operatorname{cl} T$ is bounded (on $\operatorname{cl}_{\mathcal{H}_1}(\mathcal{D}(T))$).

A linear operator T from $\mathcal{H}_1$ into $\mathcal{H}_2$ is said to be *densely defined* if its domain $\mathcal{D}(T)$ is dense in $\mathcal{H}_1$. Define

$$\mathcal{D}(T^*) = \big\{y \in \mathcal{H}_2 : \exists\, u \in H_1 \text{ s.t. } \langle Tx, y\rangle_{\mathcal{H}_2} = \langle x, u\rangle_{\mathcal{H}_1} \ \forall\, x \in \mathcal{D}(T_\mu)\big\}.$$

Since $\mathcal{D}(T)$ is dense in $\mathcal{H}_1$, the vector $u \in \mathcal{H}_1$ satisfying $\langle Tx, y\rangle_{\mathcal{H}_2} = \langle x, u\rangle_{\mathcal{H}_1}$ for all $x \in \mathcal{D}(T_\mu)$ is uniquely determined by y; put $T^* y = u$. Then T^* is a linear operator from $\mathcal{H}_2$ into $\mathcal{H}_1$ with domain $\mathcal{D}(T^*)$. The linear operator T is called the *adjoint operator* of T. Note that

$$\langle Tx, y\rangle_{\mathcal{H}_2} = \langle x, T^*y\rangle_{\mathcal{H}_1}$$

for every $x \in \mathcal{D}(T)$ and $y \in \mathcal{D}(T^*)$.

Let T be a linear operator on a Hilbert space $\mathcal{H}$, i.e., a linear operator from $\mathcal{H}$ into itself, with domain $\mathcal{D}(T)$. We say that T is *symmetric* (or *Hermitian*) if

$$\langle Tx, y\rangle_{\mathcal{H}} = \langle x, Ty\rangle_{\mathcal{H}}$$

for every $x, y \in \mathcal{D}(T)$. Note that T is symmetric if and only if $\langle Tx, x\rangle_{\mathcal{H}} \in \mathbb{R}$ for all $x \in \mathcal{D}(T)$. We say that T is *nonnegative* and write $T \geq 0$ if $\langle Tx, x\rangle_{\mathcal{H}} \geq 0$ for all $x \in \mathcal{D}(T)$.

Now suppose that T is a densely defined linear operator on $\mathcal{H}$. Then T is symmetric if and only if $T \subseteq T^*$. We say that T is *self-adjoint* if $T = T^*$, and that T is *essentially self-adjoint* if $\operatorname{cl} T$ is self-adjoint, or equivalently, if $\operatorname{cl} T = T^*$. We say that T is *hyponormal* if $\mathcal{D}(T) \subseteq \mathcal{D}(T^*)$ and $\|Tx\|_{\mathcal{H}} \geq \|T^*x\|_{\mathcal{H}}$ for all $x \in \mathcal{D}(T)$, that T is *formally normal* if $\mathcal{D}(T) \subseteq \mathcal{D}(T^*)$ and $\|Tx\|_{\mathcal{H}} = \|T^*x\|_{\mathcal{H}}$ for all $x \in \mathcal{D}(T)$, and that T is *normal* if T is formally normal and $\mathcal{D}(T) = \mathcal{D}(T^*)$.

2.6 Toeplitz and Hankel operators

A classical Toeplitz operator is the compression of a multiplication operator on $L^2(\mathbb{T})$ to $H^2(\mathbb{T})$. In the past, studies on Toeplitz operators only covered the case that symbol is a bounded function. Such Toeplitz operators are always bounded operators. A more general symbol may be taken from $L^2(\mathbb{T})$, and the corresponding Toeplitz operator is an unbounded operator.

Let $\varphi \in L^2(\mathbb{T})$. The Toeplitz operator T_φ with symbol φ is the linear operator on H^2 with domain

$$\mathcal{D}(T_\varphi) = \{f \in H^2 : P(\varphi f) \in H^2\},$$

given by

$$T_\varphi f = P(\varphi f) \qquad (f \in \mathcal{D}(T_\varphi)).$$

(Recall that every function in H^2 may be identified with its nontangential limit function which belongs to $H^2(\mathbb{T})$.) Clearly, $A(\mathbb{D}) \subseteq \mathcal{D}(T_\varphi)$. Hence T_φ is densely defined. Also, T_φ is closed. Observe that

$$\langle T_\varphi z^j, z^i \rangle_{H^2} = \langle \varphi, z^{i-j} \rangle_{L^2(\mathbb{T})} = \widehat{\varphi}(i-j)$$

for every $i, j \in \mathbb{N}_0$. Hence the matrix representation of T_φ with respect to the orthonormal basis $\{1, z, z^2, \dots\}$ is

$$\begin{bmatrix} \widehat{\varphi}(0) & \widehat{\varphi}(-1) & \widehat{\varphi}(-2) & \cdots \\ \widehat{\varphi}(1) & \widehat{\varphi}(0) & \widehat{\varphi}(-1) & \cdots \\ \widehat{\varphi}(2) & \widehat{\varphi}(1) & \widehat{\varphi}(0) & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

A matrix of this form is called a Toeplitz matrix, i.e., an infinite matrix $(\alpha_{i,j})_{i,j\in\mathbb{N}_0}$ is said to be a *Toeplitz matrix* if $\alpha_{i,j} = \alpha_{i+1,j+1}$ for every $i, j \in \mathbb{N}_0$.

The properties of a Toeplitz operator are closed related to the properties of its symbol, and vice versa. For example, T_φ is an isometry if and only if φ is a nonzero inner function. From the paper [BH64], we can show the following.

Theorem 2.1. ([BH64]) *Let $\varphi \in L^2(\mathbb{T})$. Then T_φ is bounded if and only if $\varphi \in L^\infty(\mathbb{T})$, in which case $\|T_\varphi\| = \|\varphi\|_\infty$.*

Roughly speaking, a Hankel operator is the remaining part after subtracting the Toeplitz part from a multiplication. In order to make a Hankel operator to a linear operator on H^2, we need to introduce a linear transformation with "flips" Fourier coefficients of a function appropriately. Define $J : L^1(\mathbb{T}) \to L^1(\mathbb{T})$ by

$$Jf(z) = \bar{z}f(\bar{z}) \qquad (f \in L^1(\mathbb{T})).$$

Then the mapping J is linear and isometric, $JJ = I$, and

$$\widehat{Jf}(n) = \widehat{f}(-n-1)$$

for every $n \in \mathbb{Z}$. It follows that J maps $H^1(\mathbb{T})$ onto $\overline{H^1_0(\mathbb{T})}$ and $\overline{H^1_0(\mathbb{T})}$ onto $H^1(\mathbb{T})$. As a mapping on $L^2(\mathbb{T})$, J is a unitary operator which maps $H^2(\mathbb{T})$ onto $\overline{H^2_0(\mathbb{T})}$ and $\overline{H^2_0(\mathbb{T})}$ onto $H^2(\mathbb{T})$.

Let $\varphi \in L^2(\mathbb{T})$. The Hankel operator H_φ with symbol φ is the linear operator on H^2 with domain

$$\mathcal{D}(H_\varphi) = \{f \in H^2 : PJ(\varphi f) \in H^2\},$$

given by

$$H_\varphi f = PJ(\varphi f) \qquad (f \in \mathcal{D}(H_\varphi)).$$

Then H_φ is a densely defined closed linear operator on H^2. An explicit formula for $H_\varphi f$ is

$$(H_\varphi f)(z) = \int_{\mathbb{T}} \frac{J(\varphi f)(\zeta)}{1-\bar{\zeta}z}\, dm(\zeta) = \int_{\mathbb{T}} \frac{\zeta\varphi(\zeta)f(\zeta)}{1-\zeta z}\, dm(\zeta) \qquad (z \in \mathbb{D}).$$

Since $Jz^j = \bar{z}^{j+1}$, it follows that

$$\langle H_\varphi z^j, z^i\rangle_{H^2} = \langle J(\varphi z^i), z^j\rangle_{L^2(\mathbb{T})} = \langle \varphi z^i, Jz^j\rangle_{H^2} = \widehat{\varphi}(-i-j-1)$$

for every $i, j \in \mathbb{N}_0$. Thus the matrix representation of H_φ with respect to the orthonormal basis $\{1, z, z^2, \dots\}$ is

$$\begin{bmatrix} \widehat{\varphi}(-1) & \widehat{\varphi}(-2) & \widehat{\varphi}(-3) & \cdots \\ \widehat{\varphi}(-2) & \widehat{\varphi}(-3) & \widehat{\varphi}(-4) & \cdots \\ \widehat{\varphi}(-3) & \widehat{\varphi}(-4) & \widehat{\varphi}(-5) & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

A matrix of this form is called a Hankel matrix, i.e., an infinite matrix $(\alpha_{i,j})_{i,j\in\mathbb{N}_0}$ is said to be a *Hankel matrix* if the entries $\alpha_{i,j}$ depend only on the sum $i + j$ of their two indices.

Suppose that $\varphi = \varphi_0 + \varphi_1$, where $\varphi_0 \in H^2(\mathbb{T})$ and $\varphi_1 \in \overline{H_0^2(\mathbb{T})}$. The properties of the Hankel operator H_φ are independent of φ_0. For example, H_φ is the zero operator on H^2 if and only if $\varphi_1 = 0$. Likewise, the boundedness of H_φ is determined only by φ_1. The following characterization of the bounded Hankel operators is due to Z. Nehari [Neh57].

Theorem 2.2. ([Neh57]) *Let $\varphi \in L^2(\mathbb{T})$. Then H_φ is bounded if and only if there exists a function $\psi \in L^\infty(\mathbb{T})$ such that $\widehat{\psi}(n) = \varphi(n)$ for all $n < 0$. In this case*

$$\|H_\varphi\| = \inf\big\{\|\psi\|_\infty : \widehat{\psi}(n) = \widehat{\varphi}(n),\, n < 0\big\}.$$

To deal with moment problems, we need some concepts for infinite matrices. Let $(\alpha_{ij})_{i,j\in\mathbb{N}_0}$ be an infinite matrix whose entries are complex numbers. We say that $(\alpha_{ij})_{i,j\in\mathbb{N}_0}$ is *nonnegative definite* if

$$\sum_{i,j=0}^{\infty} \alpha_{ij} x_j \overline{x_i} \geq 0$$

for every finitely supported sequence $(x_j)_{j\in\mathbb{N}_0}$ of complex numbers. We say that $(\alpha_{ij})_{i,j\in\mathbb{N}_0}$ is *positive definite* if

$$\sum_{i,j=0}^{\infty} \alpha_{ij} x_j \overline{x_i} > 0$$

for every finitely supported nonzero sequence $(x_j)_{j\in\mathbb{N}_0}$ of complex numbers.

We say that $(\alpha_{ij})_{i,j\in\mathbb{N}_0}$ is *bounded* if there exists a constant $c > 0$ such that

$$\sum_{i=0}^{\infty} \left| \sum_{j=0}^{\infty} \alpha_{ij} x_j \right|^2 \leq c \cdot \sum_{j=0}^{\infty} |x_j|^2$$

for every finitely supported sequence $(x_j)_{j\in\mathbb{N}_0}$ of complex numbers. Note that $(\alpha_{ij})_{i,j\in\mathbb{N}_0}$ is bounded if and only if $(\alpha_{ij})_{i,j\in\mathbb{N}_0}$ determines a bounded operator on ℓ^2, or equivalently, there exists a bounded operator on H^2 whose matrix representation is $(\alpha_{ij})_{i,j\in\mathbb{N}_0}$

For a sequence $s = (s_n)_{n\in\mathbb{Z}}$ of complex numbers, we denote by $T(s)$ the infinite Toeplitz matrix corresponding to s, i.e.,

$$T(s) = \begin{bmatrix} s_0 & s_{-1} & s_{-2} & \cdots \\ s_1 & s_0 & s_{-1} & \cdots \\ s_2 & s_1 & s_0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

For $n \in \mathbb{N}_0$ and for a finite sequence $s = (s_j)_{j=-n}^{n}$, we denote by $T_n(s)$ the $(n+1) \times (n+1)$ Toeplitz matrix corresponding to s, i.e.,

$$T_n(s) = \begin{bmatrix} s_0 & s_{-1} & \cdots & s_{-n} \\ s_1 & s_0 & \cdots & s_{-n+1} \\ \vdots & \vdots & \ddots & \vdots \\ s_n & s_{n-1} & \cdots & s_0 \end{bmatrix}.$$

For a sequence $s = (s_n)_{n\in\mathbb{N}_0}$ of complex numbers, we denote by $H(s)$ the infinite Hankel matrix corresponding to s, i.e.,

$$H(s) = \begin{bmatrix} s_0 & s_1 & s_2 & \cdots \\ s_1 & s_2 & s_3 & \cdots \\ s_2 & s_3 & s_4 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

For $n \in \mathbb{N}_0$ and for a finite sequence $s = (s_j)_{j=0}^n$, we denote by $H_n(s)$ the $(n+1) \times (n+1)$ Hankel matrix corresponding to s, i.e.,

$$H_n(s) = \begin{bmatrix} s_0 & s_1 & \cdots & s_n \\ s_1 & s_2 & \cdots & s_{n+1} \\ \vdots & \vdots & \ddots & \vdots \\ s_n & s_{n+1} & \cdots & s_{2n} \end{bmatrix}.$$

If $\varphi \in L^2(\mathbb{T})$, then the matrix representations of T_φ and H_φ are $T(\widehat{\varphi})$ and $H(\widehat{J\varphi})$, respectively.

Chapter 3
Toeplitz operators with symbols of measures

In this chapter, we introduce the definition of Toeplitz operators whose symbols are complex Borel measures on the unit circle. Then we study their spectral properties and the boundedness.

3.1 Toeplitz operators with symbols of measures

Let μ be a complex Borel measure on $\mathbb{T}$. For each function $f \in A(\mathbb{D})$, the measure $f \cdot \mu$ is a complex Borel measure on $\mathbb{T}$, and hence the Cauchy transform $P(f \cdot \mu)$ is an analytic function on $\mathbb{D}$. Define

$$\mathcal{D}(T_\mu) := \left\{ f \in A(\mathbb{D}) : P(f \cdot \mu) \in H^2 \right\}.$$

It is easy to show that $\mathcal{D}(T_\mu)$ is a linear subspace of H^2. Now define

$$T_\mu f := P(f \cdot \mu) \qquad (f \in \mathcal{D}(T_\mu)).$$

Then T_μ is a linear operator on H^2 with domain $\mathcal{D}(T_\mu)$.

Definition 3.1. The linear operator T_μ is called the *Toeplitz operator with symbol* μ.

The following proposition shows that the above definition is a proper generalization of the Toeplitz operators whose symbols are L^2-functions.

Proposition 3.2. *Suppose that $\mu \ll m$ and $\varphi := \frac{d\mu}{dm} \in L^2(\mathbb{T})$. Then $\mathcal{D}(T_\mu) = A(\mathbb{D})$ and*

$$T_\mu f = T_\varphi f$$

for every $f \in A(\mathbb{D})$.

Proof. Suppose that $\mu = \varphi \cdot m$, where $\varphi \in L^2(\mathbb{T})$. Let f be any function in $A(\mathbb{D})$. Then

$$P(f \cdot \mu)(z) = \int_{\mathbb{T}} \frac{f(\zeta)}{1 - \bar{\zeta} z} \, d\mu(\zeta) = \int_{\mathbb{T}} \frac{f(\zeta)\varphi(\zeta)}{1 - \bar{\zeta} z} \, dm(\zeta) = P(\varphi f)(z)$$

for every $z \in \mathbb{D}$, i.e., $P(f \cdot \mu) = P(\varphi f)$. Since $\varphi f \in L^2(\mathbb{T})$, it follows that $P(f \cdot \mu) \in H^2$. Hence $f \in \mathcal{D}(T_\mu)$ and

$$T_\mu f = P(f \cdot \mu) = P(\varphi f) = T_\varphi f.$$

This completes the proof. □

Remark 3.3.

(a) (Toeplitz operators with L^1-symbols) Every function $\varphi \in L^1(\mathbb{T})$ would be regarded as the absolutely continuous measure $\varphi \cdot m \in M(\mathbb{T})$. Hence we may use Definition 3.1 to define Toeplitz operators with L^1-symbols: If $\varphi \in L^1(\mathbb{T})$ and $\mu = \varphi \cdot m$, then

$$\mathcal{D}(T_\mu) = \left\{ f \in A(\mathbb{D}) : P(\varphi f) \in H^2 \right\}$$

and

$$T_\mu f = P(\varphi f) \qquad (f \in \mathcal{D}(T_\mu)).$$

(b) (Toeplitz operators with H^1-symbols) Let $\varphi \in H^1(\mathbb{T})$, and put $\mu = \varphi \cdot m \in M(\mathbb{T})$. For every $f \in A(\mathbb{D})$, $\varphi f \in H^1(\mathbb{T})$. Hence $P(\varphi f) = \varphi f$ (if we view φ in the right-hand side as a function in H^1). It follows that

$$\mathcal{D}(T_\mu) = \left\{ f \in A(\mathbb{D}) : \varphi f \in H^2 \right\}$$

and

$$T_\mu f = \varphi f \qquad (f \in \mathcal{D}(T_\mu)).$$

This shows that a Toeplitz operator with H^1-symbol behaves as a multiplication. Notice that the action of T_μ is same as that of T_φ defined in [Sar08, Section 5]. (In that paper, the domain of T_φ is given by $\mathcal{D}(T_\varphi) = \{f \in H^2 : \varphi f \in H^2\}$.) Moreover, since φ is of Smirnov class, $\varphi = b/a$ for some $a, b \in H^\infty$ such that a is an outer function, $a(0) > 0$, and $|a|^2 + |b|^2 = 1$ on $\mathbb{T}$. In this case, $\mathcal{D}(T_\varphi) = aH^2$ (cf. [Sar08]). It follows that

$$\mathcal{D}(T_\mu) = \mathcal{D}(T_\varphi) \cap A(\mathbb{D}) = aH^2 \cap A(\mathbb{D}).$$

Since a is an outer function, it follows that aH^2 is dense in H^2.

Let us see some concrete examples.

Example 3.4.

(a) Let φ be the analytic function on $\mathbb{D}$ such that $(\varphi(z))^2 = (1-z)^{-1}$ and $\varphi(0) = 1$. Then $\varphi \in H^1$ but $\varphi \notin H^2$. Put $\mu = \varphi \cdot m$. By Remark 3.3(b), we have

$$\mathcal{D}(T_\mu) = \left\{f \in A(\mathbb{D}) : \varphi f \in H^2\right\}.$$

How large is the domain $\mathcal{D}(T_\mu)$? Suppose that $g \in A(\mathbb{D})$ and $g(1) \neq 0$. Then there exists a constant $c > 0$ such that $|g| \geq c$ on a neighborhood of $\zeta = 1$. It follows that $\varphi g \notin H^2$. Hence $g \notin \mathcal{D}(T_\mu)$. This shows that

$$\mathcal{D}(T_\mu) \subseteq \{f \in A(\mathbb{D}) : f(1) = 0\}.$$

On the other hand, if $r > 0$ and if ψ_r is the function in $A(\mathbb{D})$ which satisfies

$(\psi_r(z))^{1/r} = 1 - z$ and $\psi_r(0) = 1$, then, for every $g \in A(\mathbb{D})$,

$$\begin{aligned}
\|\varphi\psi_r g\|_2^2 &= \int_{\mathbb{T}} |\varphi(\zeta)|^2 |\psi_r(\zeta)|^2 |g(\zeta)|^2 \, dm(\zeta) \\
&= \int_{\mathbb{T}} \frac{|1-\zeta|^{2r}}{|1-\zeta|} |g(\zeta)|^2 \, dm(\zeta) \\
&\le \|g\|_\infty^2 \cdot \int_{\mathbb{T}} |1-\zeta|^{2r-1} \, dm(\zeta) = \frac{\|g\|_\infty^2}{2\pi} \int_{-\pi}^{\pi} |1-e^{it}|^{2r-1} \, dt \\
&\le \frac{\|g\|_\infty^2}{2\pi} \int_{-\pi}^{\pi} |t|^{2r-1} \, dt = \frac{\|g\|_\infty^2}{\pi} \frac{\pi^{2r}}{2r},
\end{aligned}$$

and hence $\varphi\psi_r g \in H^2$, i.e., $\psi_r g \in \mathcal{D}(T_\mu)$. It follows that

$$\bigcap_{r>0} \psi_r A(\mathbb{D}) \subseteq \mathcal{D}(T_\mu).$$

Since $\psi_1 = 1 - z$, we have

$$(1-z)A(\mathbb{D}) \subseteq \mathcal{D}(T_\mu).$$

In particular, $\mathcal{D}(T_\mu)$ contains all polynomials vanishing at $\zeta = 1$.

(b) Let $\mu = \delta_1$ be the unit mass concentrated at $\zeta = 1$. Note that the measure μ is discrete. Observe that, for $f \in A(\mathbb{D})$,

$$P(f \cdot \mu)(z) = \int_{\mathbb{T}} \frac{f(\zeta)}{1 - \bar{\zeta} z} \, d\mu(\zeta) = \frac{f(1)}{1-z} \qquad (z \in \mathbb{D}).$$

Since $\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n$, the function $\frac{1}{1-z}$ does not belong to H^2. It follows that $P(f \cdot \mu) \in H^2$ if and only if $f(1) = 0$. Therefore

$$\mathcal{D}(T_\mu) = \big\{ f \in A(\mathbb{D}) : f(1) = 0 \big\}.$$

Also, we have

$$T_\mu f = 0$$

for all $f \in \mathcal{D}(T_\mu)$. Hence T_μ is trivial, i.e., $T_\mu f = 0$ for all $f \in \mathcal{D}(T_\mu)$. Consequently, T_μ is bounded (on $\mathcal{D}(T_\mu)$). Notice that $\mathcal{D}(T_\mu)$ does not contain the constant function 1. We show later that $\mathcal{D}(T_\mu)$ is dense in H^2.

(c) (The Cantor middle-third measure) Let φ be the Cantor function. Then φ is continuous and monotonically increasing. Hence there exists a nonnegative Borel measure μ on $\mathbb{T}$ such that

$$\mu\big(\{e^{2\pi i\theta} : 0 \leq \theta < t\}\big) = \varphi(t) \qquad (0 \leq t \leq 1).$$

The measure μ (so-called the Cantor middle-third measure) is a typical example of a singular continuous measure. We refer the reader to the paper [BM74] which treats measures of Cantor type. It is known that

$$\widehat{\mu}(n) = (-1)^n \prod_{j=1}^{\infty} \cos \frac{2\pi n}{3^j} \qquad (n \in \mathbb{Z}).$$

Hence

$$|\widehat{\mu}(n)|^2 = \prod_{j=1}^{\infty} \left(1 - \sin^2 \frac{2\pi n}{3^j}\right) \qquad (n \in \mathbb{Z}).$$

Since $0 \leq \sin^2 \frac{2\pi n}{3^j} < 1$ for each j and $\sum_{j=1}^{\infty} \sin^2 \frac{2\pi n}{3^j} < \infty$, it follows that $\widehat{\mu}(n) \neq 0$. Note also that $\widehat{\mu}(-n) = \widehat{\mu}(n)$ and $\widehat{\mu}(3n) = \widehat{\mu}(n)$ for every $n \in \mathbb{Z}$. We may here ask the following questions:

(i) What is $\mathcal{D}(T_\mu)$? Is $\mathcal{D}(T_\mu)$ dense in H^2 ?

(iii) What is T_μ ? Is T_μ trivial ?

Let us fix a complex Borel measure μ on $\mathbb{T}$. One natural question is when the domain $\mathcal{D}(T_\mu)$ of the Toeplitz operator T_μ is dense in H^2. It seems not to be easy to answer this question in general. The following lemma is used to derive some properties of $\mathcal{D}(T_\mu)$, which are also useful to determine the density of $\mathcal{D}(T_\mu)$ in H^2. Recall that S denotes the shift on $H(\mathbb{D})$.

Lemma 3.5. *For every* $\mu \in M(\mathbb{T})$ *and* $f \in A(\mathbb{D})$,

$$P(Sf \cdot \mu) = SP(f \cdot \mu) + P(Sf \cdot \mu)(0).$$

Proof. For each $z \in \mathbb{D}$,

$$\begin{aligned} P(Sf \cdot \mu)(z) - P(Sf \cdot \mu)(0) &= \int_{\mathbb{T}} \frac{\zeta f(\zeta)}{1 - \bar{\zeta} z}\, d\mu(\zeta) - \int_{\mathbb{T}} \zeta f(\zeta)\, d\mu(\zeta) \\ &= \int_{\mathbb{T}} \frac{\bar{\zeta} z}{1 - \bar{\zeta} z} \zeta f(\zeta)\, d\mu(\zeta) \\ &= z \int_{\mathbb{T}} \frac{f(\zeta)}{1 - \bar{\zeta} z}\, d\mu(\zeta) \\ &= zP(f \cdot \mu)(z) = SP(f \cdot \mu)(z). \end{aligned}$$

□

Proposition 3.6. *Let* $\mu \in M(\mathbb{T})$ *and* $\alpha \in \mathbb{C} \backslash \mathbb{T}$. *Then the following hold.*

(a) *For* $f \in A(\mathbb{D})$, $f \in \mathcal{D}(T_\mu)$ *if and only if* $(S - \alpha) f \in \mathcal{D}(T_\mu)$.

(b) *For* $f \in H^2$, $f \in \mathrm{cl}_{H^2}(\mathcal{D}(T_\mu))$ *if and only if* $(S - \alpha) f \in \mathrm{cl}_{H^2}(\mathcal{D}(T_\mu))$.

Proof. (a) Let $f \in A(\mathbb{D})$. Then, by Lemma 3.5,

$$\begin{aligned} P((S - \alpha) f \cdot \mu) &= P(Sf \cdot \mu) - P(\alpha f \cdot \mu) \\ &= SP(f \cdot \mu) + P(Sf \cdot \mu)(0) - \alpha P(f \cdot \mu) \\ &= (S - \alpha) P(f \cdot \mu) + P(Sf \cdot \mu)(0). \end{aligned}$$

Hence $P((S-\alpha) f \cdot \mu) \in H^2$ if and only if $(S-\alpha)P(f \cdot \mu) \in H^2$. Since $P(f \cdot \mu) \in H(\mathbb{D})$ and $\alpha \notin \mathbb{T}$, it follows that $P(f \cdot \mu) \in H^2$ if and only if $(S - \alpha)P(f \cdot \mu) \in H^2$. This proves (a).

(b) Suppose that $f \in \mathrm{cl}_{H^2}(\mathcal{D}(T_\mu))$. Then there exists a sequence (f_j) in $\mathcal{D}(T_\mu)$ such that $\|f - f_j\|_2 \to 0$. Since $S - \alpha$ is a bounded operator on H^2, we have

$$\big\|(S - \alpha) f - (S - \alpha) f_j\big\|_2 = \big\|(S - \alpha)(f - f_j)\big\|_2 \to 0.$$

By (a), each $(S-\alpha)f_j$ belongs to $\mathcal{D}(T_\mu)$. It follows that $(S-\alpha)f \in \mathrm{cl}_{H^2}(\mathcal{D}(T_\mu))$.

Conversely, suppose that $f \in H^2$ and $(S-\alpha)f \in \mathrm{cl}_{H^2}(\mathcal{D}(T_\mu))$. Then there exists a sequence (g_j) in $\mathcal{D}(T_\mu)$ such that

$$\big\|(S-\alpha)f - g_j\big\|_2 \to 0.$$

We want to show that $f \in \mathrm{cl}_{H^2}(\mathcal{D}(T_\mu))$. To see this we consider two cases.

Case 1: $|\alpha| < 1$. Assume first that $g_j(\alpha) = 0$ for all j. Then

$$g_j = (S-\alpha)f_j,$$

where $f_j \in A(\mathbb{D})$. Since $g_j \in \mathcal{D}(T_\mu)$, it follows from (a) that $f_j \in \mathcal{D}(T_\mu)$. Note that the approximate point spectrum of S is $\sigma_{\mathrm{ap}}(S) = \mathbb{T}$ (cf. [Hal]). Since α does not belong to $\sigma_{\mathrm{ap}}(S)$, the operator $S-\alpha$ is bounded below on H^2. It follows that there exists a constant $c > 0$ such that

$$\big\|(S-\alpha)f - g_j\big\|_2 = \big\|(S-\alpha)(f-f_j)\big\|_2 \geq c \cdot \|f - f_j\|_2$$

for all j. This implies that $\|f - f_j\|_2 \to 0$. Therefore $f \in \mathrm{cl}_{H^2}(\mathcal{D}(T_\mu))$.

In the case that $g_j(\alpha) \neq 0$ for some j, we may assume that $g_1(\alpha) \neq 0$. Note that $g_j \to (S-\alpha)f$ weakly. Hence $g_j(z) \to ((S-\alpha)f)(z)$ for each $z \in \mathbb{D}$. In particular, we have

$$g_j(\alpha) \to 0.$$

Now put

$$h_j = g_j - \frac{g_j(\alpha)}{g_1(\alpha)} g_1.$$

Then $h_j \in \mathcal{D}(T_\mu)$ and $h_j(\alpha) = 0$ for all j. Observe that

$$\big\|(S-\alpha)f - h_j\big\|_2 \leq \big\|(S-\alpha)f - g_j\big\|_2 + \left|\frac{g_j(\alpha)}{g_1(\alpha)}\right| \|g_1\|_2.$$

It follows that

$$\left\|(S-\alpha)f - h_j\right\|_2 \to 0.$$

Hence, by the preceding paragraph, we conclude that $f \in \mathrm{cl}_{H^2}(\mathcal{D}(T_\mu))$.

Case 2: $|\alpha| > 1$. The operator $S - \alpha$ is invertible. Hence $(S-\alpha)^{-1}g_j \to f$ in H^2. Since $\mathcal{D}(T_\mu)$ is S-invariant by (a), each $(S-\alpha)^{-1}g_j$ belongs to $\mathcal{D}(T_\mu)$. It follows that $f \in \mathrm{cl}_{H^2}(\mathcal{D}(T_\mu))$. □

Remark 3.7.

(a) If we take $\alpha = 0$ in Proposition 3.6, it follows that the linear subspaces $\mathcal{D}(T_\mu)$ and $\mathrm{cl}_{H^2}(\mathcal{D}(T_\mu))$ are S-invariant.

(b) The identity in Lemma 3.5 can be rewritten as

$$S^* P(Sf \cdot \mu) = P(f \cdot \mu).$$

Consequently, we have $S^* T_\mu S f = T_\mu f$ for every $f \in \mathcal{D}(T_\mu)$.

Proposition 3.8. *Let $\mu \in M(\mathbb{T})$. Then one of the following holds:*

(i) $\mathcal{D}(T_\mu) = \{0\}$.

(ii) $\mathcal{D}(T_\mu)$ *is dense in* H^2.

(iii) $\mathrm{cl}_{H^2}(\mathcal{D}(T_\mu)) = \theta H^2$, *where θ is a singular inner function.*

Proof. By Proposition 3.6, $\mathrm{cl}_{H^2}(\mathcal{D}(T_\mu))$ is a S-invariant subspace of H^2. It follows from Beurling's theorem that

$$\mathrm{cl}_{H^2}(\mathcal{D}(T_\mu)) = \theta H^2$$

for some inner function θ. If $\theta = 0$, then the case (i) occurs. If θ is a nonzero constant function, the case (ii) occurs. Now suppose that θ is nonconstant. We show that θ

has no zero in $\mathbb{D}$. To see this, choose a nonzero function f in $\mathcal{D}(T_\mu)$. Fix an arbitrary point α of $\mathbb{D}$, and let n be the multiplicity of the zero of f at α. Then

$$f(z) = (z-\alpha)^n g(z) \qquad (z \in \mathbb{D}),$$

where $g \in H^2$ and $g(\alpha) \neq 0$. Hence, by a repeated application of Proposition 3.6(a), we have $g \in \mathcal{D}(T_\mu)$. It follows that $g = \theta h$ for some $h \in H^2$. Therefore $\theta(\alpha) \neq 0$. Since α was arbitrary, we conclude that θ is a singular inner function. $\square$

The following proposition gives a sufficient condition for the domain $\mathcal{D}(T_\mu)$ to be dense in H^2.

Proposition 3.9. *If* $\mathrm{cl}_{H^2}(\mathcal{D}(T_\mu))$ *contains a polynomial, then* $\mathcal{D}(T_\mu)$ *is dense in* H^2.

Proof. Suppose that $\mathrm{cl}_{H^2}(\mathcal{D}(T_\mu))$ contains a polynomial. Then, by Proposition 3.6(b), there exists a polynomial $p \in \mathrm{cl}_{H^2}(\mathcal{D}(T_\mu))$, all of whose zeros are in $\mathbb{T}$, such that $p(0) = 1$. Let $\zeta_1, \ldots, \zeta_N \in \mathbb{T}$ be the zeros of p, listed according to their multiplicities. Then

$$p(z) = (1-\overline{\zeta_1}z)\cdots(1-\overline{\zeta_N}z).$$

Choose a sequence (k_n) in $\mathbb{N}$ such that $k_{n+1} > Nk_n$ (e.g., $k_n = (N+1)^n$). For each $n \in \mathbb{N}$, define

$$p_n(z) = \frac{1}{n}\sum_{j=1}^{n} \bigl(1-(\overline{\zeta_1}z)^{k_j}\bigr)\cdots\bigl(1-(\overline{\zeta_N}z)^{k_j}\bigr).$$

All of them are polynomials, divisible by p. Since $\mathrm{cl}_{H^2}(\mathcal{D}(T_\mu))$ is S-invariant, the polynomials p_n belong to $\mathcal{D}(T_\mu)$. It follows by a direct computation that

$$\|1-p_n\|_2^2 \leq \frac{n}{n^2}\left[\binom{N}{1}\binom{N}{1} + \cdots + \binom{N}{N}\binom{N}{N}\right]$$

for every $n \in \mathbb{N}$. This implies that $f_n \to 1$ in H^2. Therefore, the constant function 1 belongs to $\mathrm{cl}_{H^2}(\mathcal{D}(T_\mu))$. Since $\mathrm{cl}_{H^2}(\mathcal{D}(T_\mu))$ is S-invariant, we conclude that $\mathrm{cl}_{H^2}(\mathcal{D}(T_\mu)) = H^2$, in other words, $\mathcal{D}(T_\mu)$ is dense in H^2. $\square$

Remark 3.10. Proposition 3.9 shows that the domains $\mathcal{D}(T_\mu)$, presented in (a) and (b) of Example 3.4, are dense in H^2, because they contain the polynomial $p(z) = 1-z$. The proof of Proposition 3.9 shows that every polynomial all of whose zeros are in $\mathbb{T}$ is an outer function.

In order to consider the matrix representation of a linear operator on H^2, it is necessary that its domain contains all polynomials. Let us interpret the condition that $\mathcal{D}(T_\mu)$ contains all polynomials. Note that this is equivalent to the condition that $\mathcal{D}(T_\mu)$ contains a polynomial which does not vanish on $\mathbb{T}$, by Proposition 3.6(a).

Proposition 3.11. *Let $\mu \in M(\mathbb{T})$. Then the following are equivalent:*

(i) *$\mathcal{D}(T_\mu)$ contains all polynomials, or equivalently, $\mathcal{D}(T_\mu)$ contains the constant function* 1*.*

(ii) *$\mu \ll m$ and $\frac{d\mu}{dm} \in H^2(\mathbb{T}) + \overline{H^1_0(\mathbb{T})}$.*

Proof. (i)⇒(ii): Suppose that the constant function 1 belongs to $\mathcal{D}(T_\mu)$. Then $P\mu = P(1 \cdot \mu) \in H^2$. Let ψ denote the nontangential limit function of $P\mu$. Since $P\mu = \sum_{n=0}^\infty \widehat{\mu}(n) z^n$, it follows that $\widehat{\psi}(n) = \widehat{\mu}(n)$ for all $n \in \mathbb{N}_0$. Put $\nu = \mu - \psi \cdot m$. Then $\nu \in M(\mathbb{T})$ and

$$\widehat{\nu}(n) = \widehat{\mu}(n) - \widehat{\psi}(n) = 0$$

for all $n \in \mathbb{N}_0$. It follows from the F. and M. Riesz theorem that $\nu \ll m$ and $\nu = \chi \cdot m$ for some $\chi \in \overline{H^1_0(\mathbb{T})}$. Thus we have $\mu = \nu + \psi \cdot m = (\chi + \psi) \cdot m$. This proves (ii).

(ii)⇒(i): Suppose that (ii) holds so that $\mu = (\psi + \chi) \cdot m$ for some $\psi \in H^2(\mathbb{T})$ and $\chi \in \overline{H^1_0(\mathbb{T})}$. Then $\widehat{\mu}(n) = \widehat{\psi}(n)$ for all $n \in \mathbb{N}_0$. Hence we have

$$\sum_{n=0}^\infty |\widehat{\mu}(n)|^2 < \infty.$$

Since $P\mu = \sum_{n=0}^{\infty} \widehat{\mu}(n) z^n$, it follows that $P(1 \cdot \mu) = P\mu \in H^2$. Clearly, the constant function 1 belongs to $A(\mathbb{D})$. Therefore $1 \in \mathcal{D}(T_\mu)$. $\square$

Corollary 3.12. *Let $\mu \in M(\mathbb{T})$ be a real measure. Then $\mathcal{D}(T_\mu) = A(\mathbb{D})$ if and only if $\mu \ll m$ and $\frac{d\mu}{dm} \in L^2(\mathbb{T})$.*

Proof. Suppose that $\mathcal{D}(T_\mu) = A(\mathbb{D})$. Then $\mu \ll m$ and $\mu = (\psi + \chi) \cdot m$ for some $\psi \in H^2(\mathbb{T})$ and $\chi \in \overline{H_0^1(\mathbb{T})}$ by Proposition 3.11. Since μ is a real measure, we have

$$\widehat{\mu}(-n) = \int_{\mathbb{T}} \overline{z}^{-n}\, d\mu = \overline{\int_{\mathbb{T}} \overline{z}^{n}\, d\mu} = \overline{\widehat{\mu}(n)}$$

for every $n \in \mathbb{Z}$. Thus $\widehat{\chi}(-n) = \overline{\widehat{\psi}(n)}$ for every $n \in \mathbb{N}$. Since $\psi \in H^2(\mathbb{T})$, we have

$$\sum_{n=-\infty}^{-1} |\widehat{\chi}(n)|^2 = \sum_{n=1}^{\infty} |\widehat{\psi}(n)|^2 < \infty.$$

It follows that $\chi \in \overline{H_0^2(\mathbb{T})}$. Therefore $\frac{d\mu}{dm} = \psi + \chi \in L^2(\mathbb{T})$.

The converse is a part of Proposition 3.2 $\square$

Remark 3.13. One question is to ask whether Corollary 3.12 holds for any complex measure $\mu \in M(\mathbb{T})$. If $\mu \ll m$ and $\frac{d\mu}{dm} \in L^2(\mathbb{T})$, then $\mathcal{D}(T_\mu) = A(\mathbb{D})$ by Proposition 3.2. Conversely, suppose that $\mathcal{D}(T_\mu) = A(\mathbb{D})$. Then $\mu \ll m$ and $\mu = (\psi + \chi) \cdot m$ for some $\psi \in H^2(\mathbb{T})$ and $\chi \in \overline{H_0^1(\mathbb{T})}$ by Proposition 3.11. Hence, for every $f \in A(\mathbb{D})$,

$$P(f \cdot \mu) = P((\psi + \chi) f) = P(\psi f) + P(\chi f).$$

Since $P(f \cdot \mu) \in H^2$ and $P(\psi f) \in H^2$, it follows that $P(\chi f) \in H^2$. This shows that

$$\mathcal{D}(T_{\chi \cdot m}) = A(\mathbb{D}).$$

Thus, the question is eventually to ask whether χ belongs to $\overline{H_0^2(\mathbb{T})}$ or not.

The following conjecture arises naturally:

Conjecture 3.14. *Every Toeplitz operator with singular symbol is trivial.*

We give a partial answer by using a known fact about the Cauchy transform. Let E be a closed subset of $\mathbb{T}$, and let

$$F(H^2, E) = \{g \in H^2 : g = P\mu \text{ for some } \mu \in M(E)\}.$$

It is known that $F(H^2, E) = \{0\}$ if and only if $m(E) = 0$ (cf. [CMR, Theorem 5.5.2]).

Proposition 3.15. *If $\mu \in M(\mathbb{T})$ is singular and $m(\operatorname{supp}\mu) = 0$, then T_μ is trivial.*

Proof. Let $E := \operatorname{supp}\mu$. By assumpsion $m(E) = 0$. Hence $F(H^2, E) = \{0\}$. Suppose that $f \in \mathcal{D}(T_\mu)$, i.e., $P(f \cdot \mu) \in H^2$. Note that $\operatorname{supp}(f \cdot \mu) \subseteq \operatorname{supp}\mu = E$. Hence $f \cdot \mu \in M(E)$. So the function $P(f \cdot \mu) \in H^2$ belongs to $F(H^2, E) = \{0\}$. It follows that $P(f \cdot \mu) = 0$. We have shown that $P(f \cdot \mu) \in H^2$ implies $P(f \cdot \mu) = 0$. In other words,

$$f \in \mathcal{D}(T_\mu) \implies Tf = 0.$$

Hence T_μ is trivial (on its domain). □

The Cantor-middle third measure μ in Example 3.4(c) is a singular continuous measure and its support is the Cantor set (in $\mathbb{T}$) whose Lebesgue measure is 0. Hence Proposition 3.15 implies that T_μ is trivial.

We have seen that the Toeplitz operator T_μ in Example 3.4(b) is a densely defined trivial linear operator. This result can be extended to the case that μ has a finite support. In this case, the fact that T_μ is trivial may follow from Proposition 3.15. However, we give a direct proof and also find the domain $\mathcal{D}(T_\mu)$.

Proposition 3.16. *Let $\mu \in M(\mathbb{T})$ be a discrete measure whose support is a finite set. Then the Toeplitz operator T_μ is a densely defined trivial linear operator.*

Proof. Suppose that $\zeta_1, \ldots, \zeta_N$ are distinct points of $\mathbb{T}$, and that

$$\mu = c_1\delta_{\zeta_1} + \cdots + c_N\delta_{\zeta_N}, \tag{3.1.1}$$

where $c_1, \ldots, c_N$ are nonzero complex numbers.

We first show that

$$\mathcal{D}(T_\mu) = \big\{f \in A(\mathbb{D}) : f(\zeta_1) = \cdots = f(\zeta_N) = 0\big\}. \tag{3.1.2}$$

For any $f \in A(\mathbb{D})$,

$$P(f \cdot \mu)(z) = \sum_{j=1}^{N} c_j P(f \cdot \delta_{\zeta_j})(z) = \sum_{j=1}^{N} \frac{c_j f(\zeta_j)}{1 - \overline{\zeta_j} z} \qquad (z \in \mathbb{D}). \tag{3.1.3}$$

It follows that

$$\big\{f \in A(\mathbb{D}) : f(\zeta_1) = \cdots = f(\zeta_N) = 0\big\} \subseteq \mathcal{D}(T_\mu).$$

Conversely, let $f \in \mathcal{D}(T_\mu)$. Then $P(f \cdot \mu) \in H^2$. For each j, put

$$F_j(\zeta) = \frac{c_j f(\zeta_j)}{1 - \overline{\zeta_j}\zeta} \qquad (\zeta \in \mathbb{T}).$$

Then $F = \sum_{j=1}^{N} F_j$ is the nontangential limit function of $P(f \cdot \mu)$. Thus $F \in H^2(\mathbb{T})$. Choose disjoint open arcs $I_j \subseteq \mathbb{T}$ with $\zeta_j \in I_j$. If we fix an index j_0, then $\chi_{I_{j_0}} \cdot F \in L^2(\mathbb{T})$. Also, $\chi_{I_{j_0}} \cdot F_j \in L^\infty(\mathbb{T})$ for each $j \neq j_0$. Hence

$$\chi_{I_{j_0}} \cdot F_{j_0} = \chi_{I_{j_0}} \cdot F - \sum_{j \neq j_0} (\chi_{I_{j_0}} \cdot F_j) \in L^2(\mathbb{T}).$$

Since $\chi_{\mathbb{T} \setminus I_{j_0}} \cdot F_{j_0} \in L^\infty(\mathbb{T})$, it follows that

$$F_{j_0} = \chi_{I_{j_0}} \cdot F_{j_0} + \chi_{\mathbb{T} \setminus I_{j_0}} \cdot F_{j_0} \in L^2(\mathbb{T}).$$

This implies that $f(\zeta_{j_0}) = 0$, because otherwise $F_{j_0} \notin L^2(\mathbb{T})$ by Parseval's theorem. Since j_0 was arbitrary, we have $f(\zeta_j) = 0$ for each j. It follows that

$$\mathcal{D}(T_\mu) \subseteq \big\{ f \in A(\mathbb{D}) : f(\zeta_1) = \cdots = f(\zeta_N) = 0 \big\}.$$

This proves (3.1.2). In particular, $\mathcal{D}(T_\mu)$ contains the polynomial

$$p(z) = (z - \zeta_1) \cdots (z - \zeta_N)$$

. Hence, by Proposition 3.9, $\mathcal{D}(T_\mu)$ is dense in H^2.

Equations (3.1.2) and (3.1.3) shows that $T_\mu f = 0$ for all $f \in \mathcal{D}(T_\mu)$, i.e., T_μ is trivial. $\square$

To each Toeplitz operator T_μ there corresponds an infinite Toeplitz matrix $T(\widehat{\mu})$. In general, however, it is a bit awkward to call $T(\widehat{\mu})$ as the matrix representation of T_μ, because the domain $\mathcal{D}(T_\mu)$ may not contains the monomials z^n. Nevertheless, often, information about T_μ gives information about $T(\widehat{\mu})$. The following is one of examples.

Corollary 3.17. *Let $\mu \in M(\mathbb{T})$ be a discrete measure whose support consists of N points of $\mathbb{T}$. Then $\det T_n(\widehat{\mu}) = 0$ for all $n \geq N$.*

Proof. Suppose that μ is the discrete measure given by (3.1.1). Then the domain $\mathcal{D}(T_\mu)$ is given by (3.1.2). Choose a polynomial p in $\mathcal{D}(T_\mu)$ whose degree is N (e.g., $p(z) = (z - \zeta_1) \cdots (z - \zeta_N)$). Write $p = \sum_{k=0}^N a_k z^k$. Since $T_\mu z^k = \sum_{n=0}^\infty \widehat{\mu}(n-k) z^n$, it follows that

$$\begin{aligned} 0 = T_\mu p = \sum_{k=0}^N a_k T_\mu z^k &= \sum_{k=0}^N a_k \sum_{n=0}^\infty \widehat{\mu}(n-k) z^n \\ &= \sum_{n=0}^\infty \left(\sum_{k=0}^\infty a_k \widehat{\mu}(n-k) \right) z^n. \end{aligned}$$

Hence we have

$$\sum_{k=0}^{N} a_k \widehat{\mu}(n-k) = 0 \tag{3.1.4}$$

for all $n \in \mathbb{N}_0$. Now let $n \geq N$, and put

$$x = \begin{bmatrix} a_0 & \cdots & a_N & 0 & \cdots & 0 \end{bmatrix}^T \in \mathbb{C}^{n+1}$$

Then, by (3.1.4), $T_n(\widehat{\mu})x = 0$, i.e., $x \in \ker T_n(\widehat{\mu})$. Since $x \neq 0$, the square matrix $T_n(\widehat{\mu})$ is not invertible, or equivalently, $\det T_n(\widehat{\mu}) = 0$. $\square$

Remark 3.18. Corollary 3.17 can also be shown by a simple manipulation for the matrix $T_n(\widehat{\mu})$:

$$T_n(\widehat{\mu}) = \begin{bmatrix} \sum_{k=1}^N c_k & \sum_{k=1}^N c_k\zeta_k & \cdots & \sum_{k=1}^N c_k\zeta_k^n \\ \sum_{k=1}^N c_k\bar{\zeta}_k & \sum_{k=1}^N c_k & \cdots & \sum_{k=1}^N c_k\zeta_k^{n-1} \\ \vdots & \vdots & \ddots & \vdots \\ \sum_{k=1}^N c_k\bar{\zeta}_k^n & \sum_{k=1}^N c_k\bar{\zeta}_k^{n-1} & \cdots & \sum_{k=1}^N c_k \end{bmatrix}$$

$$= \begin{bmatrix} 1 & \cdots & 1 \\ \bar{\zeta}_1 & \cdots & \bar{\zeta}_N \\ \vdots & \ddots & \vdots \\ \bar{\zeta}_1^n & \cdots & \bar{\zeta}_N^n \end{bmatrix} \begin{bmatrix} c_1 & 0 & \cdots & 0 \\ 0 & c_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & c_N \end{bmatrix} \begin{bmatrix} 1 & \zeta_1 & \cdots & \zeta_1^n \\ \vdots & \vdots & \ddots & \vdots \\ 1 & \zeta_N & \cdots & \zeta_N^n \end{bmatrix}$$

for each $n \in \mathbb{N}_0$. This proves Corollary 3.17, as well as

$$\det T_{N-1}(\widehat{\mu}) = \prod_{1 \leq i < j \leq N} |\zeta_j - \zeta_i|^2 \prod_{k=1}^{N} c_k \neq 0.$$

Consequently we obtain

$$\operatorname{rank} T_n(\widehat{\mu}) = N$$

for every $n \geq N$.

3.2 The spectral properties of T_μ

In this section, we investigate some spectral properties of Toeplitz operators T_μ. The following lemma is the key to answer questions about spectral properties of T_μ.

Lemma 3.19. *Let $\mu \in M(\mathbb{T})$. Then*

$$\langle T_\mu f, g\rangle = \int_{\mathbb{T}} f\overline{g}\, d\mu \tag{3.2.1}$$

for every $f \in \mathcal{D}(T_\mu)$ and $g \in A(\mathbb{D})$.

Proof. Suppose that $f \in \mathcal{D}(T_\mu)$ and $g \in A(\mathbb{D})$. Then $T_\mu f \in H^2$. Write $T_\mu f = \sum_{n=0}^\infty a_n z^n$ and $g = \sum_{n=0}^\infty b_n z^n$. Then

$$\langle T_\mu f, g\rangle = \sum_{n=0}^\infty a_n \overline{b_n}.$$

Observe that, for each $z \in \mathbb{D}$,

$$\begin{aligned}(T_\mu f)(z) &= \int_{\mathbb{T}} \frac{f(\zeta)}{1-\bar{\zeta}z}\, d\mu(\zeta) = \int_{\mathbb{T}} f(\zeta) \sum_{n=0}^\infty \bar{\zeta}^n z^n\, d\mu(\zeta)\\ &= \sum_{n=0}^\infty \left[\int_{\mathbb{T}} f(\zeta)\bar{\zeta}^n\, d\mu(\zeta)\right] z^n.\end{aligned}$$

Hence we have

$$a_n = \int_{\mathbb{T}} f(\zeta)\bar{\zeta}^n\, d\mu(\zeta).$$

Observe that, for each $0 < r < 1$,

$$g_r = \sum_{n=0}^\infty b_n r^n z^n \in A(\mathbb{D}).$$

It follows that

$$\begin{aligned}\langle T_\mu f, g_r\rangle = \sum_{n=0}^\infty a_n \overline{b_n} r^n &= \sum_{n=0}^\infty \int_{\mathbb{T}} f(\zeta)\bar{\zeta}^n \overline{b_n} r^n\, d\mu(\zeta)\\ &= \int_{\mathbb{T}} f(\zeta)\overline{\sum_{n=0}^\infty b_n r^n \zeta^n}\, d\mu(\zeta)\\ &= \int_{\mathbb{T}} f\overline{g_r}\, d\mu.\end{aligned}$$

If we let $r \to 1$, then $\|g - g_r\|_\infty \to 0$, and hence $\langle T_\mu, g_r\rangle \to \langle T_\mu, g\rangle$ and $\int_{\mathbb{T}} f\overline{g_r}\, d\mu \to \int_{\mathbb{T}} f\overline{g}\, d\mu$. This proves (3.2.1). $\square$

What is the adjoint of T_μ? Assume that $\mu \in M(\mathbb{T})$ and $\mathcal{D}(T_\mu)$ is dense in H^2. Then the adjoint T_μ^* of T_μ can be defined; the domain of T_μ^* is

$$\mathcal{D}(T_\mu^*) = \big\{g \in H^2 : \exists\, h \in H^2 \text{ s.t. } \langle T_\mu f, g\rangle = \langle f, h\rangle \;\forall\, f \in \mathcal{D}(T_\mu)\big\},$$

and, for each $g \in \mathcal{D}(T_\mu^*)$, $T_\mu^* g$ is the (unique) element of H^2 such that

$$\langle T_\mu f, g\rangle = \langle f, T_\mu^* g\rangle$$

for every $f \in \mathcal{D}(T_\mu)$.

If $\varphi \in L^\infty(\mathbb{T})$, then $T_\varphi^* = T_{\overline{\varphi}}$. Hence it is reasonable to expect that the adjoint of T_μ is the Toeplitz operator whose symbol is the "complex conjugation" of μ. For $\mu \in M(\mathbb{T})$, define

$$\overline{\mu}(E) = \overline{\mu(E)} \qquad (E \in \mathscr{B}_{\mathbb{T}}).$$

Then $\overline{\mu} \in M(\mathbb{T})$. Of course, $\mu \in M(\mathbb{T})$ is a real measure if and only if $\overline{\mu} = \mu$. Note that

$$\widehat{\overline{\mu}}(n) = \overline{\widehat{\mu}(-n)}$$

for every $n \in \mathbb{Z}$.

Proposition 3.20. *Let $\mu \in M(\mathbb{T})$. Assume that $\mathcal{D}(T_\mu)$ is dense in H^2. Then*

$$T_{\overline{\mu}} \subseteq T_\mu^*.$$

Proof. Let $g \in \mathcal{D}(T_{\overline{\mu}})$. By Proposition 3.19, it follows that

$$\langle T_\mu f, g\rangle = \int_{\mathbb{T}} f\overline{g}\, d\mu = \overline{\int_{\mathbb{T}} g\overline{f}\, d\overline{\mu}} = \langle f, T_{\overline{\mu}} g\rangle$$

for every $f \in \mathcal{D}(T_\mu)$. It follows that $g \in \mathcal{D}(T_\mu^*)$ and $T_\mu^* g = T_{\bar{\mu}} g$. Therefore we conclude that

$$\mathcal{D}(T_{\bar{\mu}}) \subseteq \mathcal{D}(T_\mu^*),$$

and $T_\mu^* g = T_{\bar{\mu}} g$ for every $g \in \mathcal{D}(T_{\bar{\mu}})$. This completes the proof. □

Remark 3.21. Let $\mu \in M(\mathbb{T})$, and let T be the restriction of the Toeplitz operator T_μ to $\mathrm{cl}_{H^2}(\mathcal{D}(T_\mu))$. Then T is a densely defined linear operator. In this case, T^* is a linear operator from H^2 onto $\mathrm{cl}_{H^2}(\mathcal{D}(T_\mu))$. By the same argument as the proof of Proposition 3.20, we have $\mathcal{D}(T_{\bar{\mu}}) \subseteq \mathcal{D}(T^*)$ and $T^* g = T_{\bar{\mu}} g$ for $g \in \mathcal{D}(T_{\bar{\mu}})$.

Now, we focus on nonnegative measures.

Proposition 3.22. *Let $\mu \in M(\mathbb{T})$ be nonnegative. Then the following hold:*

(a) *T_μ is nonnegative, i.e., $\langle T_\mu f, f \rangle \geq 0$ for all $f \in \mathcal{D}(T_\mu)$.*

(b) $\ker T_\mu = \{ f \in A(\mathbb{D}) : f = 0$ *on* $\operatorname{supp} \mu \}$.

Proof. (a) Suppose that $\mu \in M(\mathbb{T})$ is nonnegative. Then, by Lemma 3.19, we have

$$\langle T_\mu f, f \rangle = \int_{\mathbb{T}} |f|^2 \, d\mu \geq 0$$

for every $f \in \mathcal{D}(T_\mu)$.

(b) Suppose that $\mu \in M(\mathbb{T})$ is nonnegative. If $f \in \ker T_\mu$, then

$$\int_{\mathbb{T}} |f|^2 \, d\mu = \langle T_\mu f, f \rangle = 0.$$

Hence $f = 0$ μ-a.e. on $\mathbb{T}$. We show that $f = 0$ on $\operatorname{supp} \mu$. Assume to the contrary that $f(\zeta_0) \neq 0$ for some $\zeta_0 \in \operatorname{supp} \mu$. Since $f \in A(\mathbb{D})$, there exist a constant $\epsilon > 0$ and an open arc $I \subseteq \mathbb{T}$ with center ζ_0 such that $|f(\zeta)| \geq \epsilon$ for all $\zeta \in I$. Since $\zeta_0 \in \operatorname{supp} \mu$, I must have positive measure. It follows that

$$\int_{\mathbb{T}} |f|^2 \, d\mu \geq \int_I |f|^2 \, d\mu \geq \epsilon \cdot \mu(I) > 0,$$

which is a contradiction. Therefore $f(\zeta) = 0$ for all $\zeta \in \operatorname{supp} \mu$.

We have shown that

$$\ker T_\mu \subseteq \{f \in A(\mathbb{D}) : f = 0 \text{ on } \operatorname{supp} \mu\}.$$

The reverse inclusion is trivial. $\square$

Remark 3.23. T_μ may be nonnegative even though μ is complex. For example, for any complex number α, the Toeplitz operator $T_{\alpha \cdot \delta_1}$ is trivial, and hence it is nonnegative.

3.3 The boundedness of T_μ

In this section, we investigate the boundedness of Toeplitz operators T_μ. We give a characterization of the $\mathbb{T}$-Carleson measures in terms of the boundedness of Toeplitz operators (Theorem 3.28). We first consider the case that T_μ is nonnegative.

Proposition 3.24. *Let $\mu \in M(\mathbb{T})$. Suppose that T_μ is a nonnegative densely defined linear operator. Then T_μ is bounded if and only if there exists a constant $c > 0$ such that*

$$\int_{\mathbb{T}} |f|^2 \, d\mu \leq c \cdot \|f\|_2^2 \tag{3.3.1}$$

for every $f \in \mathcal{D}(T_\mu)$.

Proof. Suppose that T_μ is bounded. Then there exists a constant $c > 0$ such that $\|T_\mu f\|_2 \leq c \cdot \|f\|_2$, and hence that

$$\int_{\mathbb{T}} |f|^2 \, d\mu = \langle T_\mu f, f \rangle \leq \|T_\mu f\|_2 \cdot \|f\|_2 \leq c \cdot \|f\|_2^2,$$

for every $f \in \mathcal{D}(T_\mu)$.

Conversely, suppose that there exists a constant $c > 0$ such that (3.3.1) holds, i.e.,

$$\langle T_\mu f, f\rangle \le c \cdot \|f\|_2^2,$$

for every $f \in \mathcal{D}(T_\mu)$. For each $g, h \in \mathcal{D}(T_\mu)$ with $\|g\|_2 = \|h\|_2 = 1$, observe that

$$\langle T_\mu(g+h), g+h\rangle - \langle T_\mu(g-h), g-h\rangle = 4\,\mathrm{Re}\langle T_\mu g, h\rangle.$$

By assumption, we have

$$4\,\mathrm{Re}\langle T_\mu g, h\rangle \le c\big(\|g+h\|_2^2 + \|g-h\|_2^2\big) = 2c\big(\|g\|_2^2 + \|h\|_2^2\big) = 4c$$

whenever $g, h \in \mathcal{D}(T_\mu)$ and $\|g\|_2 = \|h\|_2 = 1$. Now choose $\zeta_0 \in \mathbb{T}$ so that

$$\big|\langle T_\mu g, h\rangle\big| = \zeta_0 \cdot \langle T_\mu g, h\rangle.$$

Then

$$\big|\langle T_\mu g, h\rangle\big| = \langle T_\mu(\zeta_0 g), h\rangle = \mathrm{Re}\langle T_\mu(\zeta_0 g), h\rangle \le c.$$

Since $\mathcal{D}(T_\mu)$ is dense in H^2, it follows that

$$\|T_\mu g\|_2 = \sup\big\{\big|\langle T_\mu g, h\rangle\big| : h \in \mathcal{D}(T_\mu),\ \|h\|_2 = 1\big\} \le c.$$

This shows that T_μ is bounded (on $\mathcal{D}(T_\mu)$). $\square$

Remark 3.25.

(a) A nonnegative measure $\mu \in M(\mathbb{T})$ is $\mathbb{T}$-Carleson if and only if there exists a constant $c > 0$ such that the inequality (3.3.1) holds for every $f \in A(\mathbb{D})$. Hence every nonnegative $\mathbb{T}$-Carleson measure derives a bounded Toeplitz operator.

(b) If $\mu \in M(\mathbb{T})$ is nonnegative and T_μ is bounded, does it follows that μ is $\mathbb{T}$-Carleson? The answer to this question is negative in general, even if $\mathcal{D}(T_\mu)$ is dense in H^2. For example, consider the unit mass δ_1 concentrated at $\zeta = 1$. As we have seen in Example 3.4(b), T_{δ_1} is densely defined and bounded. But δ_1 is not $\mathbb{T}$-Carleson, because, for the polynomials

$$p_n = \frac{1}{\sqrt{n}}(1 + \cdots + z^{n-1}),$$

$\|p_n\|_2 = 1$ and $p_n(1) = \sqrt{n}$. Notice that $\mathcal{D}(T_{\delta_1})$ does not contain all polynomials.

We will show that the last statement in Remark 3.25(a) is still true for any complex $\mathbb{T}$-Carleson measure. In other words, every $\mathbb{T}$-Carleson measure induces a bounded Toeplitz operator.

Proposition 3.26. *If $\mu \in M(\mathbb{T})$ is a $\mathbb{T}$-Carleson measure, then $\mathcal{D}(T_\mu) = A(\mathbb{D})$ and T_μ is bounded.*

Proof. Suppose that $\mu \in M(\mathbb{T})$ is $\mathbb{T}$-Carleson. Then there exists a contant $c > 0$ such that

$$\int_{\mathbb{T}} |f|^2 \, d|\mu| \leq c \cdot \|f\|_2^2$$

for every $f \in A(\mathbb{D})$. It follows that

$$\left| \int_{\mathbb{T}} f\overline{g} \, d\mu \right| \leq c \cdot \|f\|_2 \cdot \|g\|_2$$

for every $f, g \in A(\mathbb{D})$. It follows that the sesquilinear form $\phi(f,g) = \int_{\mathbb{T}} f\overline{g}\, d\mu$ is bounded. Hence there exists a bounded operator T on H^2 (with domain H^2) such that

$$\langle Tf, g\rangle = \int_{\mathbb{T}} f\overline{g} \, d\mu \tag{3.3.2}$$

whenever $f, g \in A(\mathbb{D})$. Now fix $f \in A(\mathbb{D})$. Since $z^n \in A(\mathbb{D})$, (3.3.2) gives

$$\widehat{Tf}(n) = \langle Tf, z^n\rangle = \int_{\mathbb{T}} f(\zeta)\overline{\zeta}^n \, d\mu(\zeta)$$

for every $n \in \mathbb{N}_0$. Hence, for each $z \in \mathbb{D}$,

$$\begin{aligned}
Tf(z) &= \sum_{n=0}^{\infty} \widehat{Tf}(n) z^n = \sum_{n=0}^{\infty} \int_{\mathbb{T}} f(\zeta)\overline{\zeta}^n z^n \, d\mu(\zeta) \\
&= \int_{\mathbb{T}} f(\zeta) \sum_{n=0}^{\infty} \overline{\zeta}^n z^n \, d\mu(\zeta) \\
&= \int_{\mathbb{T}} \frac{f(\zeta)}{1-\overline{\zeta}z} \, d\mu(\zeta) = P(f \cdot \mu)(z).
\end{aligned}$$

Since $Tf \in H^2$, we have $f \in \mathcal{D}(T_\mu)$ and $T_\mu f = Tf$. We conclude that $\mathcal{D}(T_\mu) = A(\mathbb{D})$ and $T_\mu f = Tf$ for every $f \in \mathcal{D}(T_\mu)$. Since T is bounded, so is T_μ. This completes the proof. □

For the converse of Proposition 3.26 we show the following:

Proposition 3.27. *Let $\mu \in M(\mathbb{T})$. Suppose that $\mathcal{D}(T_\mu)$ contains all polynomials and T_μ is bounded. Then $\mu \ll m$ and $\frac{d\mu}{dm} \in L^\infty(\mathbb{T})$.*

Proof. Since $\mathcal{D}(T_\mu)$ contains all polynomials, it follows from Proposition 3.11 that

$$\mu = (\chi + \psi) \cdot m$$

for some $\chi \in \overline{H_0^1(\mathbb{T})}$ and $\psi \in H^2(\mathbb{T})$. Since T_μ is bounded, there exists a constant $c > 0$ such that

$$\|T_\mu f\|_2 \le c \cdot \|f\|_2$$

for every $f \in \mathcal{D}(T_\mu)$.

Now let $k \in \mathbb{N}_0$. Then $z^k \in \mathcal{D}(T_\mu)$ and

$$\widehat{(z^k \cdot \mu)}(n) = \int_{\mathbb{T}} \overline{\zeta}^n \zeta^k \, d\mu(\zeta) = \widehat{\mu}(n-k) \qquad (n \in \mathbb{Z}).$$

Hence we have

$$T_\mu z^k = P(z^k \cdot \mu) = \sum_{n=0}^{\infty} \widehat{(z^k \cdot \mu)}(n) z^n = \sum_{n=0}^{\infty} \widehat{\mu}(n-k) z^n \qquad (z \in \mathbb{D}).$$

Since $T_\mu z^k \in H^2$, it follows that

$$\|T_\mu z^k\|_2^2 = \sum_{n=0}^{\infty} |\widehat{\mu}(n-k)|^2 = \sum_{n=-k}^{-1} |\widehat{\chi}(n)|^2 + \sum_{n=0}^{\infty} |\widehat{\psi}(n)|^2 = \sum_{n=-k}^{-1} |\widehat{\chi}(n)|^2 + \|\psi\|_2^2.$$

Since $z^k \in \mathcal{D}(T_\mu)$ and $\|z^k\|_2 = 1$, we have

$$\|T_\mu z^k\|_2 \le c \cdot \|z^k\|_2 = c.$$

Hence

$$\sum_{n=-k}^{-1} |\widehat{\chi}(n)|^2 = \|T_\mu z^k\|_2^2 - \|\psi\|_2^2 \le c^2 - \|\psi\|_2^2.$$

Since k was arbitrary, it follows that $\chi \in L^2(\mathbb{T})$. Put $\varphi = \chi + \psi$. Then $\varphi \in L^2(\mathbb{T})$ and $\mu = \varphi \cdot m$. From Proposition 3.2 we obtain that $\mathcal{D}(T_\mu) = A(\mathbb{D})$ and

$$T_\mu f = T_\varphi f$$

for all $f \in A(\mathbb{D})$. Recall that the domain of T_φ is $\mathcal{D}(T_\varphi) = \{f \in H^2 : P(\varphi f) \in H^2\}$. Since $A(\mathbb{D})$ is dense in $\mathcal{D}(T_\varphi)$, the boundedness of T_μ on $A(\mathbb{D})$ implies the boundedness of T_φ on $\mathcal{D}(T_\varphi)$. Hence Theorem 2.1 implies that $\varphi \in L^\infty(\mathbb{T})$. Since $\varphi = \frac{d\mu}{dm}$, the proof is complete. $\square$

The main theorem of this section now follows.

Theorem 3.28. *Let $\mu \in M(\mathbb{T})$. Then the following statements are equivalent:*

(i) *μ is a $\mathbb{T}$-Carleson measure.*

(ii) *$\mu \ll m$ and $\frac{d\mu}{dm} \in L^\infty(\mathbb{T})$.*

(iii) *$\mathcal{D}(T_\mu)$ contains all polynomials and T_μ is bounded.*

If these conditions are satisfied and if $\varphi = \frac{d\mu}{dm}$, then $\mathcal{D}(T_\mu) = A(\mathbb{D})$ and

$$T_\mu f = T_\varphi f$$

for every $f \in A(\mathbb{D})$.

Proof. (i)⇒(iii) by Proposition 3.26 and (iii)⇒(ii) by Proposition 3.27. The last assertion follows from Proposition 3.2. Therefore it only remains to prove the implication (ii)⇒(i).

Suppose that $\mu = \varphi \cdot m$, where $\varphi \in L^\infty(\mathbb{T})$. Then $|\mu| = |\varphi| \cdot m$. It follows that

$$\int_{\mathbb{T}} |f|^2 \, d|\mu| = \int_{\mathbb{T}} |f|^2 |\varphi| \, dm \leq \|\varphi\|_\infty \|f\|_2^2$$

for every $f \in A(\mathbb{D})$. Hence μ is $\mathbb{T}$-Carleson and the proof is complete. □

Remark 3.29. We have established a one-to-one correspondence between essentially bounded measurable functions and $\mathbb{T}$-Carleson measures: The set of all $\mathbb{T}$-Carleson measures on $\mathbb{T}$ is

$$M_{\mathrm{ac}}(\mathbb{T}) = \big\{\varphi \cdot m : \varphi \in L^\infty(\mathbb{T})\big\}.$$

3.4 The moment problem and T_μ

For a (complex) Borel measure μ supported on a set X, moments of μ are given as the integrals of monomials z^n over X with respect to μ, i.e.,

$$\int_X z^n \, d\mu.$$

Hence to each measure μ there corresponds a sequence consisting of moments of μ. The moment problem is the inverse problem of finding a measure whose moment sequence matches a given sequence. We refer the reader to the text [Sch2] which treats the various moment problems in great detail. In this section we focus on two moment problems, namely the trigonometric moment problem and the Hamburger moment problem, which are related to Toeplitz and Hankel matrices.

Let $s = (s_n)_{n\in\mathbb{N}_0}$ be a sequence of complex numbers. The classical *Hamburger moment problem* is to find a nonnegative measure supported on the real line $\mathbb{R}$ which represents the sequence s, in the sense that

$$\int_{\mathbb{R}} |x|^n \, d\mu(x) < \infty$$

and

$$s_n = \int_{\mathbb{R}} x^n \, d\mu(x)$$

for every $n \in \mathbb{N}_0$. The following solution of the Hamburger moment problem is due to H. Hamburger [Ham20]: There exists a nonnegative Borel measure μ on $\mathbb{R}$ which represents the sequence s if and only if the infinite Hankel matrix $H(s)$ is nonnegative definite.

It truns out that the boundedness of the Hankel matrix $H(s)$ is related to a kind of "compatibility" of the representing measure. Indeed, for a sequence $s = (s_n)_{n\in\mathbb{N}_0}$ of complex numbers, the following statements hold (cf. [Pel], [Wid66], [Pow]):

(a) The matrix $H(s)$ is nonnegative semidefinite and bounded [compact] if and only if there exists a nonnegative Carleson measure [vanishing Carleson measure] μ supported on $(-1, 1)$ such that $s_n = \int_{-1}^{1} x^n \, d\mu(x)$ for every $n \in \mathbb{N}_0$.

(b) The matrix $H(s)$ is bounded [compact] if and only if there exists a complex Carleson measure [vanishing Carleson measure] μ such that $s_n = \int_{\mathbb{D}} z^n \, d\mu(z)$ for every $n \in \mathbb{N}_0$.

Next we review the moment problem related to Toeplitz matrices. Let $s = (s_n)_{n\in\mathbb{Z}}$ be a sequence of complex numbers. The classical *trigonometric moment problem* is to find a nonnegative measure on the unit circle $\mathbb{T}$ which represents the sequence s, in the sense that

$$s_n = \int_{\mathbb{T}} \overline{\zeta}^n \, d\mu(\zeta) \tag{3.4.1}$$

for every $n \in \mathbb{Z}$. Note that the moment sequence of μ is $\widehat{\mu}$ in this problem. The solution of the trigonometric moment problem is originally due to O. Toeplitz [Toe11]: There exists a nonnegative Borel measure μ on $\mathbb{T}$ which represents the sequence s if and only if the infinite Toeplitz matrix $T(s)$ is nonnegative definite.

In view of the statements (a) and (b) in the preceding paragraph, we may ask the following.

Question 3.30. Let $s = (s_n)_{n\in\mathbb{Z}}$ be a sequence of complex numbers and $\theta \in H^\infty$ be an inner function. When does there exist a Carleson measure μ for $\mathcal{H}(\theta)$ which represents the sequence s, i.e., satisfies (3.4.1) for all $n \in \mathbb{Z}$?

We give an answer to Question 3.30 for the case $\theta = 0$.

Proposition 3.31. *For a sequence $s = (s_n)_{n\in\mathbb{Z}}$ of complex numbers, the following are equivalent:*

(i) *There exists a $\mathbb{T}$-Carleson measure μ which represents the sequence s, i.e.,*

$$s_n = \int_{\mathbb{T}} \overline{\zeta}^n \, d\mu(\zeta)$$

for all $n \in \mathbb{Z}$.

(ii) *There exists a function $\varphi \in L^\infty(\mathbb{T})$ such that $s_n = \widehat{\varphi}(n)$ for all $n \in \mathbb{Z}$.*

(iii) *The infinite Toeplitz matrix $T(s)$ is bounded.*

Proof. (i)$\Leftrightarrow$(ii): This follows from Theorem 2.1.

(ii)$\Rightarrow$(iii): Let $\varphi \in L^\infty$ be such that $s_n = \widehat{\varphi}(n)$ for all $n \in \mathbb{Z}$. Then the Toeplitz operator T_φ is bounded. Since $T(s)$ is the matrix representation of T_φ, it follows that the matrix $T(s)$ is bounded.

(iii)$\Rightarrow$(ii): Suppose that the matrix $T(s)$ is bounded. Then there exists a bounded operator T on H^2 whose matrix representation is $T(s)$. If the matrix representation of a bounded operator on H^2 is a Toeplitz matrix, it is a Toeplitz operator with L^∞-symbol (cf. [BH64, Theorem 4]). It follows that $T = T_\varphi$ for some $\varphi \in L^\infty$. The matrix representation of T_φ is $T(\widehat{\varphi})$. Therefore $s_n = \widehat{\varphi}(n)$ for all $n \in \mathbb{Z}$. $\square$

Remark 3.32. Suppose that $\mu \in M(\mathbb{T})$ is absolutel continuous, so that $\mu = \varphi \cdot m$, where $\varphi \in L^1(\mathbb{T})$. Then $\mu \geq 0$ if and only if $\varphi \geq 0$ a.e. Hence, for a sequence $s = (s_n)_{n\in\mathbb{Z}}$, the following are equivalent:

(i) There exists a nonnegative Carleson measure μ measure μ on $\mathbb{T}$ which represents the sequence s.

(ii) The infinite Toeplitz matrix $T(s)$ is bounded and nonnegative definite.

We now turn to the *truncated moment problem.* Let $s = (s_j)_{j=-n}^{n}$ be a finite sequence of complex numbers. The *truncated trigonometric moment problem* asks for nonnegative Borel measures on $\mathbb{T}$ for which

$$s_j = \int_{\mathbb{T}} \overline{\zeta}^j \, d\mu(\zeta)$$

for $j = -n, \ldots, n$. A necessary and sufficient condition that the sequence s is the moment sequence of a nonnegative Borel measure on $\mathbb{T}$ is that $T_n(s)$ is nonnegative definite (cf. [Sch2, Chapter 11]). What if the nonnegativeness of measures is replaced by the compatibility? The question for $\mathbb{T}$-Carleson measures is quite easy to answer if no further assumption is added:

For any given finite sequence $s = (s_j)_{j=-n}^{n}$ of complex numbers, put

$$\varphi(z) = \sum_{j=-n}^{n} s_n z^n,$$

and put $\mu = \varphi \cdot m$. Since $\varphi \in L^\infty(\mathbb{T})$, the measure μ is $\mathbb{T}$-Carleson. Clearly, $s_j = \widehat{\mu}(j)$ for all $j = -n, \ldots, n$. Hence any finite sequence is a part of the moment of a $\mathbb{T}$-Carleson measure.

Question 3.33. When does there exist a $\mathbb{T}$-Carleson measure $\mu \in M(\mathbb{T})$ with $\|\mu\| \leq 1$ (or $\|\frac{d\mu}{dm}\|_\infty \leq 1$) which represents the given sequence $s = (s_j)_{j=-n}^n$?

One trivial sufficient condition for s to be represented by $\mu \in M(\mathbb{T})$ with $\|\mu\| \leq 1$ is that $\sum_{j=-n}^n |s_j| \leq 1$: Put $\varphi = \sum_{j=-n}^n s_n z^n$ and $\mu = \varphi \cdot m$. Then

$$\|\mu\| = \|\varphi\|_1 \leq \|\varphi\|_\infty \leq 1.$$

Of course, that is not a necessary condition. For example, consider $s_0 = \frac{1}{4}$ and $s_1 = s_2 = s_{-1} = s_{-2} = -\frac{1}{4}$. Then $\sum |s_j| = \frac{5}{4} > 1$, but supremum on $\mathbb{T}$ of the function

$$\varphi = s_{-2}\bar{z}^2 + s_{-1}\bar{z} + s_0 + s_1 z + s_2 z^2 = \frac{1}{4}(1 - 2\cos\theta - 2\cos 2\theta)$$

occurs at $\cos\theta = -\frac{1}{4}$; $\|\varphi\|_\infty = \varphi(e^{i\theta_0}) = \frac{13}{16} < 1$. Also, $\|\varphi\|_1 \leq \|\varphi\|_2 = \frac{\sqrt{5}}{4}$.

Question 3.34. When does there exist a positive $\mathbb{T}$-Carleson measure $\mu \in M(\mathbb{T})$ which represents the given sequence $s = (s_j)_{j=-n}^n$?

Without loss of generality we assume that $s_0 = 1$; desired representing measure is a probability measure. By the known solution of truncated trigonometric moment problem, it is necessary that $T_n(s)$ is nonnegative definite.

3.5 Truncated Toeplitz operators with symbols of measures

D. Sarason has done the systematic study of truncated Toeplitz operators. We review some notions from his paper [Sar07]:

Fix a nonconstant inner function $\theta \in H^\infty$. Recall that

$$\mathcal{H}(\theta) := H^2 \ominus \theta H^2$$

and

$$\mathcal{H}_A(\theta) := \mathcal{H}(\theta) \cap A(\mathbb{D}).$$

A. B. Aleksandrov has shown that $\mathcal{H}_{A(\mathbb{D})}(\theta)$ is dense in $\mathcal{H}(\theta)$ (cf. [Alek95]). Let $P_\theta = P - M_\theta P M_{\bar{\theta}}$ be the orthogonal projection of L^2 onto $\mathcal{H}(\theta)$. The truncated Toeplitz operator A_φ on $\mathcal{H}(\theta)$ with symbol $\varphi \in L^2$ is defined by

$$A_\varphi f = P_\theta(\varphi f) \qquad (f \in \mathcal{H}_A(\theta)).$$

Recall that a comlex measure $\mu \in M(\mathbb{T})$ is called a Carleson measure for $\mathcal{H}(\theta)$ if $\mathcal{H}_A(\theta)$ is boundedly embedded in $L^2(|\mu|)$, i.e., there is a constant $c > 0$ such that

$$\int_{\mathbb{T}} |f|^2 \, d|\mu| \leq c \cdot \|f\|_2^2$$

for every $f \in \mathcal{H}_A(\theta)$.

Sarason have shown that to each Carleson measure μ for $\mathcal{H}(\theta)$ there corresponds a bounded truncated toeplitz operator A_μ which represents μ ([Sar07]): If μ is a Carleson measure for $\mathcal{H}(\theta)$, then there exists a bounded operator A_μ on $\mathcal{H}(\theta)$ such that

$$\langle A_\mu f, g \rangle = \int_{\mathbb{T}} f \bar{g} \, d\mu$$

for every $f, g \in \mathcal{H}_A(\theta)$. The operator A_μ is a bounded truncated Toeplitz operator on $\mathcal{H}(\theta)$. The converse of this has been proved by A. Baranov, R. Bessonov, and V. Kapustin [BBK11] in 2011: For every bounded truncated Toeplitz operator A on $\mathcal{H}(\theta)$, there is a Carleson measure $\mu \in M(\mathbb{T})$ for $\mathcal{H}(\theta)$ such that $A = A_\mu$.

We now turn to our attention to truncated Toeplitz operators whose symbols are given by measures. For $\mu \in M(\mathbb{T})$, define

$$P_\theta \mu := P\mu - \theta P(\bar{\theta} \cdot \mu)$$

whenever $\theta^* \in L^1(\mu)$, and define

$$\mathcal{D}_\mu^\theta \equiv \mathcal{D}(T_\mu^\theta) := \{f \in \mathcal{H}_A(\theta) : \overline{\theta^*} f \in L^1(|\mu|),\, P_\theta(f \cdot \mu) \in \mathcal{H}(\theta)\}.$$

Note that $\mathcal{D}_\mu^\theta$ is linear submanifold of H^2.

Definition 3.35. Let $\mu \in M(\mathbb{T})$. The *truncated Toeplitz operator with symbol* μ is the linear operator T_μ^θ on $\mathcal{H}(\theta)$ with domain $\mathcal{D}_\mu^\theta$, defined by

$$T_\mu^\theta f = P_\theta(f \cdot \mu) \qquad (f \in \mathcal{D}_\mu^\theta).$$

Firstly, we assume that $\mathcal{H}(\theta)$ is finite dimensional. Note that $\mathcal{H}(\theta)$ is finite dimensional if and only if θ is a finite Blaschke product.

Proposition 3.36. *Assume that* $\dim \mathcal{H}(\theta) < \infty$. *Then every* $\mu \in M(\mathbb{T})$ *is a Carleson measure for* $\mathcal{H}(\theta)$.

Proof. Let $\{e_1, \ldots, e_n\}$ be an orthonormal basis for $\mathcal{H}(\theta)$. Let $f = \sum_{i=1}^n c_i e_i \in \mathcal{H}(\theta)$. Then

$$\|f\|_2^2 = \sum_{i=1}^n |c_i|^2.$$

By the Schwarz inequality, $|\sum_{i=1}^n c_i e_i|^2 \leq n^2 \sum_{i=1}^n |c_i e_i|^2$. It follows that, for any complex measure $\mu \in M(\mathbb{T})$,

$$\begin{aligned}\int_{\mathbb{T}} |f|^2 \, d|\mu| &= \int_{\mathbb{T}} \left| \sum_{i=1}^n c_i e_i \right|^2 d|\mu| \\ &\leq n^2 \sum_{i=1}^n |c_i|^2 \int_{\mathbb{T}} |e_i|^2 \, d|\mu| \leq c \cdot \|f\|_2^2,\end{aligned}$$

where $c = n^2 \cdot \max_i \int_{\mathbb{T}} |e_i|^2 \, d|\mu|$. Therefore every $\mu \in M(\mathbb{T})$ is a $\mathbb{T}$-Carleson measure for $\mathcal{H}(\theta)$. $\square$

Proposition 3.37. *Assume that* $\dim \mathcal{H}(\theta) < \infty$. *Then* $T_\mu^\theta = A_\mu$ *for every* $\mu \in M(\mathbb{T})$.

Proof. Let $\{e_1, \ldots, e_n\}$ be an orthonormal basis for $\mathcal{H}(\theta)$. Since $\theta \in C(\overline{\mathbb{D}})$, the functions

$$k_\zeta^\theta(z) = \frac{1 - \overline{\theta(\zeta)}\theta(z)}{\bar{\zeta} z} \quad (\zeta \in \mathbb{T},\, z \in \mathbb{D})$$

is a kernel function in $\mathcal{H}(\theta)$. Hence $k_\zeta^\theta = \sum_{i=1}^n \overline{e_i(\zeta)} e_i$. Thus, for $f \in \mathcal{H}(\theta)$ and $z \in \mathbb{D}$,

$$\begin{aligned} T_\mu^\theta f(z) &= \int_{\mathbb{T}} k_\zeta^\theta(z) f(\zeta) \, d\mu(\zeta) \\ &= \sum_{i=1}^n \left[\int_{\mathbb{T}} f(\zeta) \overline{e_i(\zeta)} \, d\mu(\zeta) \right] e_i(z) \in \mathcal{H}(\theta). \end{aligned}$$

For every $f, g \in \mathcal{H}(\theta)$, we have

$$\begin{aligned} \langle T_\mu^\theta f, g \rangle &= \int_{\mathbb{T}} \left[\int_{\mathbb{T}} f(\zeta) \overline{k_z^\theta(\zeta)} \, d\mu(\zeta) \right] \overline{g(z)} \, dm(z) \\ &= \int_{\mathbb{T}} f(\zeta) \left[\int_{\mathbb{T}} k_\zeta^\theta(z) \overline{g(z)} \, dm(z) \right] d\mu(\zeta) \\ &= \int_{\mathbb{T}} f \bar{g} \, d\mu = \langle A_\mu f, g \rangle. \end{aligned}$$

We conclude that $T_\mu^\theta = A_\mu$. $\square$

Remark 3.38. If $\dim \mathcal{H}(\theta) < \infty$, then every $\mu \in M(\mathbb{T})$ is a Carleson measure for $\mathcal{H}(\theta)$ and every T_μ^θ is a bounded operator on $\mathcal{H}(\theta)$. Hence we can say that

$$T_\mu^\theta \text{ is a bounded operator on } \mathcal{H}(\theta) \iff \mu \text{ is a Carleson measure for } \mathcal{H}(\theta). \quad (3.5.1)$$

Chapter 4
The cases of measures on $\overline{\mathbb{D}}$

In this chapter we consider Borel measures on the closed unit disc $\overline{\mathbb{D}}$.

4.1 Toeplitz operators with symbols of measures on $\overline{\mathbb{D}}$

We denote the space of complex Borel measures on $\overline{\mathbb{D}}$ by $M(\overline{\mathbb{D}})$. Let μ be a complex Borel measure on $\overline{\mathbb{D}}$. For $n, k \in \mathbb{N}_0$, define the (n,k)-moment of μ by

$$\mu_{n,k} = \int_{\overline{\mathbb{D}}} z^n \bar{z}^k \, d\mu(z).$$

If $k = 0$, we simply write

$$\mu_n = \mu_{n,0} = \int_{\overline{\mathbb{D}}} z^n \, d\mu(z).$$

Observe that

$$|\mu_{n,k}| \leq \int_{\overline{\mathbb{D}}} |z|^{n+k} \, d|\mu|(z) \leq \|\mu\|.$$

Hence the double sequence $(\mu_{n,k})$ is bounded. A complex Borel measure μ on $\overline{\mathbb{D}}$ is completely determined by its moments. To see this suppose that μ and ν be complex Borel measures on $\overline{\mathbb{D}}$ such that $\mu_{n,k} = \nu_{n,k}$ for every $n, k \in \mathbb{N}_0$. Then

$$\int_{\overline{\mathbb{D}}} f \, d\mu = \int_{\overline{\mathbb{D}}} f \, d\nu$$

whenever $f = f(z, \bar{z})$ is a trigonometric polynomial. Since the trigonometric polynomials are dense in $C(\overline{\mathbb{D}})$ with respect to the supremum norm, the identity holds for every $f \in C(\overline{\mathbb{D}})$. In view of the Riesz representation theorem, this shows that the measure $\mu - \nu$ is a linear functional on $C(\overline{\mathbb{D}})$ which is zero. It follows that $\mu - \nu$ is the zero measure, i.e., $\mu = \nu$.

Let m_2 be the normalized Lebesgue measure on $\overline{\mathbb{D}}$ so that $m_2(\overline{\mathbb{D}}) = 1$. Then, for every $n, k \in \mathbb{N}_0$,

$$(m_2)_{n,k} = \int_{\mathbb{D}} z^n \bar{z}^k \, dm_2(z) = \frac{1}{\pi} \int_0^1 \int_0^{2\pi} r^{n+k+1} e^{i(n-k)} \, d\theta dr = \begin{cases} \frac{1}{n+1} & (n = k), \\ 0 & (n \neq k). \end{cases}$$

On the other hand, the moments of the unit mass δ_0 concentrated at the point $z = 0$ is

$$(\delta_0)_{n,k} = \begin{cases} 1 & (n = k = 0), \\ 0 & (\text{otherwise}). \end{cases}$$

For $f \in A(\mathbb{D})$, define the function $\mathcal{T}_\mu f$ on $\mathbb{D}$ by

$$\mathcal{T}_\mu f(z) := \int_{\overline{\mathbb{D}}} \frac{f(w)}{1 - \overline{w} z} \, d\mu(w) \qquad (z \in \mathbb{D}).$$

Note that, for each $z \in \mathbb{D}$, the series $\frac{1}{1-\overline{w}z} = \sum_{n=0}^{\infty} \overline{w}^n z^n$ converges uniformly on $\overline{\mathbb{D}}$. It follows that

$$\begin{aligned} \mathcal{T}_\mu f(z) &= \int_{\overline{\mathbb{D}}} f(w) \sum_{n=0}^{\infty} \overline{w}^n z^n \, d\mu(w) \\ &= \sum_{n=0}^{\infty} \int_{\overline{\mathbb{D}}} f(w) \overline{w}^n \, d\mu(w) z^n = \sum_{n=0}^{\infty} (f \cdot \mu)_{0,n} z^n \qquad (z \in \mathbb{D}). \end{aligned}$$

Therefore $\mathcal{T}_\mu f$ is analytic in $\mathbb{D}$. Now define

$$\mathcal{D}(\mathcal{T}_\mu) = \{ f \in A(\mathbb{D}) : \mathcal{T}_\mu f \in H^2 \}.$$

It is clear that $\mathcal{D}(\mathcal{T}_\mu)$ is a linear subspace of H^2. The mapping $\mathcal{T}_\mu$ is a linear operator on H^2 with domain $\mathcal{D}(\mathcal{T}_\mu)$.

Definition 4.1. The linear operator $\mathcal{T}_\mu$ is called the *Toeplitz operator with symbol* μ.

Observe that $\mathcal{T}_\mu$ is the operator T_μ defined in Chapter 3 when μ is supported in $\mathbb{T}$. In the case $\operatorname{supp}\mu \not\subseteq \mathbb{T}$, however, the corresponding matrix of $\mathcal{T}_\mu$ may not be a Toeplitz matrix:

$$\begin{bmatrix} \mu_{0,0} & \mu_{1,0} & \mu_{2,0} & \cdots \\ \mu_{0,1} & \mu_{1,1} & \mu_{2,1} & \cdots \\ \mu_{0,2} & \mu_{1,2} & \mu_{2,2} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}. \tag{4.1.1}$$

Remark 4.2. If the support of μ is contained in $\mathbb{T}$, then

$$\mu_{n,k} = \int_{\mathbb{T}} z^n \bar{z}^k \, d\mu(z) = \int_{\mathbb{T}} z^{n-k} \, d\mu(z) \qquad (n, k \in \mathbb{N}_0).$$

Hence the matrix (4.1.1) is a Toeplitz matrix. On the other hand, if the support of μ is contained in $(-1, 1)$, then

$$\mu_{n,k} = \int_{(-1,1)} x^n \bar{x}^k \, d\mu(x) = \int_{(-1,1)} x^{n+k} \, d\mu(x) = \mu_{n+k} \qquad (n, k \in \mathbb{N}_0).$$

Hence the matrix (4.1.1) is a Hankel matrix.

If μ is a complex Borel measure on $\overline{\mathbb{D}}$, we may write $\mu = \mu_1 + \mu_2$, where μ_1 and μ_2 are complex Borel measures on $\overline{\mathbb{D}}$ which are concentrated on $\mathbb{T}$ and $\mathbb{D}$, respectively. Then $\mathcal{T}_\mu f = T_{\mu_1} f + \mathcal{T}_{\mu_2} f$ for $f \in \mathcal{D}(T_{\mu_1})$. We have already looked at a various properties of Toeplitz operators with symbols of measure on $\mathbb{T}$. Thus in the remainder of this section we will focus on the case of measures concentrated in $\mathbb{D}$

Example 4.3.

(a) Suppose that $\alpha \in \mathbb{D}$. Let $\mu = \delta_\alpha$ be the unit mass concentrated at the point $\alpha \in \mathbb{D}$. If $f \in A(\mathbb{D})$, then

$$\mathcal{T}_\mu f(z) = \int_{\mathbb{D}} \frac{f(w)}{1 - \bar{w}z} \, d\mu(w) = \frac{f(\alpha)}{1 - \bar{\alpha}z} \qquad (z \in \mathbb{D}).$$

Hence $\mathcal{D}(\mathcal{T}_\mu) = A(\mathbb{D})$ and

$$\mathcal{T}_\mu f = \langle f, k_\alpha \rangle k_\alpha = (k_\alpha \otimes k_\alpha) f \qquad (f \in A(\mathbb{D})),$$

where $k_\alpha(z) = \frac{1}{1-\bar{\alpha}z}$ is the reproducing kernel function for H^2. Thus $\mathcal{T}_\mu$ is a restriction of the rank one projection $k_\alpha \otimes k_\alpha$ to $A(\mathbb{D})$. The matrix representation of $\mathcal{T}_\mu$ is

$$\begin{bmatrix} 1 & \alpha & \alpha^2 & \cdots \\ \bar{\alpha} & \bar{\alpha}\alpha & \bar{\alpha}\alpha^2 & \cdots \\ \bar{\alpha}^2 & \bar{\alpha}^2\alpha & \bar{\alpha}^2\alpha^2 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

(b) Consider the function

$$\varphi(x) = \frac{1}{2\sqrt{1-x}} \qquad (0 \le x < 1).$$

Let m_1 denote the Lebesgue measure on $[0,1)$. Since

$$\int_{[0,1)} |\varphi| \, dm_1 = \int_0^1 \frac{1}{2\sqrt{1-x}} \, dx = \int_0^1 \frac{1}{2\sqrt{y}} \, dy = 1,$$

the function φ belongs to $L^1(m_1)$. Hence $\mu := \varphi \cdot m_1$ is a finite nonnegative Borel measure on $\mathbb{D}$. For each $n \in \mathbb{N}_0$,

$$\mu_n = \int_0^1 \frac{x^n}{2\sqrt{1-x}} \, dx = \int_0^1 \frac{(1-y)^n}{2\sqrt{y}} \, dy = \int_0^1 (1-x^2)^n \, dx.$$

If $n \ge 1$, by integration by parts,

$$\mu_n = 2n \int_0^1 x^2(1-x^2)^{n-1} \, dx = 2n \int_0^1 (1-(1-x^2))(1-x^2)^{n-1} \, dx = 2n(\mu_{n-1} - \mu_n).$$

Hence we have

$$\mu_0 = 1, \quad \mu_n = \frac{2n}{2n+1} \mu_{n-1} \qquad (n = 1, 2, 3, \dots)$$

Using the induction, we can show that

$$\frac{1}{2n+1} \leq \mu_n^2 \leq \frac{1}{n+1}$$

for every $n \in \mathbb{N}_0$. Hence $(\mu_n) \notin \ell^2$.

On the other hand, the domain $\mathcal{D}(\mathcal{T}_\mu)$ does not contain all polynomials. Indeed, if $f_n(z) = z^n$, then

$$\mathcal{T}_\mu f_n(z) = \int_0^1 \frac{\varphi(x)x^n}{1-xz}\,d\mu(x) = \sum_{k=0}^\infty \int_0^1 \varphi(x)x^{n+k}\,d\mu(x)z^k = \sum_{k=0}^\infty \mu_{n+k}z^k,$$

which does not belong to H^2 because $(\mu_{n+k}) \notin \ell^2$. Hence $z^n \notin \mathcal{D}(\mathcal{T}_\mu)$ for every $n \in \mathbb{N}_0$. On the other hand, if $p_n(z) = 1 - z^n$, then

$$\mathcal{T}_\mu p_n(x) = \sum_{k=0}^\infty (\mu_k - \mu_{n+k})z^k.$$

Since $\mu_k - \mu_{n+k} \leq \frac{\mu_k}{2k}$, the sequence $(\mu_k - \mu_{n+k})$ belongs to ℓ^2. Hence $\mathcal{T}_\mu p_n \in H^2$, i.e., $p_n \in \mathcal{D}(\mathcal{T}_\mu)$. Observe that $\|p_n\|_2^2 = 2$, but

$$\|\mathcal{T}_\mu p_n\|_2^2 = \sum_{k=0}^\infty |\mu_k - \mu_{n+k}|^2 \to \infty$$

as $n \to \infty$. This shows that $\mathcal{T}_\mu$ is unbounded.

To consider the boundedness of $\mathcal{T}_\mu$, we first observe the following lemma.

Lemma 4.4. *Let $\mu \in M(\mathbb{D})$. Then*

$$\langle \mathcal{T}_\mu f, g\rangle = \int_{\mathbb{D}} f\bar{g}\,d\mu \tag{4.1.2}$$

for every $f \in \mathcal{D}(\mathcal{T}_\mu)$ and $g \in A(\mathbb{D})$.

Proof. Same as the proof of Lemma 3.19. □

We then have:

Theorem 4.5. *Let μ be a nonnegative finite Borel measure on $\mathbb{D}$. Then the following statements are equivalent:*

(i) *μ is a Carleson measure.*

(ii) *$\mathcal{T}_\mu$ is densely defined and bounded on its domain.*

Proof. (i)$\Rightarrow$(ii). Suppose that μ is a Carleson measure. Then there exists a constant $c > 0$ such that

$$\int_{\mathbb{D}} |fg|\, d\mu \leq c\|f\|_2\|g\|_2 \qquad (f, g \in A(\mathbb{D})).$$

Let $f := z_n$. Then

$$\mathcal{T}_\mu f(z) = \int_{\mathbb{D}} \frac{w^n}{1 - \overline{w}z}\, d\mu(w) = \sum_{j=0}^{\infty} \int_{\mathbb{D}} w^n \overline{w}^j\, d\mu(w) z^j = \sum_{j=0}^{\infty} \mu_{n,j} z^j.$$

For each $k \in \mathbb{N}_0$, put $p_k = \sum_{j=0}^{k} \mu_{n,j} z^j$. Then

$$\int_{\mathbb{D}} f\overline{p_k}\, d\mu = \int_{\mathbb{D}} z^n \sum_{j=0}^{k} \overline{\mu_{n,j}}\overline{z}^j\, d\mu(z) = \sum_{j=0}^{k} \overline{\mu_{n,j}}\mu_{n,j} = \sum_{j=0}^{k} |\mu_{n,j}|^2 = \|p_k\|_2^2.$$

Since $|\int_{\mathbb{D}} f\overline{p_k}\, d\mu| \leq c\|f\|_2\|p_k\|_2$, it follows that $\|p_k\|_2 \leq c\|f\|_2$. Hence

$$\|\mathcal{T}_\mu f\|_2^2 = \sum_{j=0}^{\infty} |\mu_{n,j}|^2 = \lim_{k\to\infty} \|p_k\|_2^2 \leq c\|f\|_2 < \infty.$$

Therefore, $\mathcal{T}_\mu f \in H^2$, i.e., $f \in \mathcal{D}(\mathcal{T}_\mu)$. We have shown that $\mathcal{D}(\mathcal{T}_\mu)$ contains every monomial z^n. Since $\mathcal{D}(\mathcal{T}_\mu)$ is a linear space, it contains all polynomials. Hence $\mathcal{D}(\mathcal{T}_\mu)$ is dense in H^2.

On the other hand, for the boundedness of $\mathcal{T}_\mu$, let $f \in \mathcal{D}(\mathcal{T}_\mu)$. Then, for each $g \in A(\mathbb{D})$,

$$|\langle \mathcal{T}_\mu f, g\rangle| = |\int_{\mathbb{D}} f\overline{g}\, d\mu| \leq \int_{\mathbb{D}} |f\overline{g}|\, d\mu \leq c\|f\|_2\|g\|_2.$$

Since $A(\mathbb{D})$ is dense in H^2, it follows that $\|\mathcal{T}_\mu f\|_2 \leq c\|f\|_2$. Hence $\mathcal{T}_\mu$ is bounded on its domain $\mathcal{D}(\mathcal{T}_\mu)$.

(ii)$\Rightarrow$(i). Suppose that $\mathcal{D}(\mathcal{T}_\mu)$ is dense in H^2 and $\mathcal{T}_\mu$ is bounded on $\mathcal{D}(\mathcal{T}_\mu)$. For every $f \in \mathcal{D}(\mathcal{T}_\mu)$,

$$\int_{\mathbb{D}} |f|^2 \, d\mu = |\langle \mathcal{T}_\mu f, f \rangle| \leq \|\mathcal{T}_\mu\| \|f\|_2^2.$$

Define $I_\mu : \mathcal{D}(\mathcal{T}_\mu) \to L^2(\mathbb{D}, \mu)$ by $I_\mu f = f$ for $f \in \mathcal{D}(\mathcal{T}_\mu)$. We may extend I_μ to a bounded operator on H^2 with bound $\|\mathcal{T}_\mu\|$. Then, for every $f \in H^2$, we have

$$\int_{\mathbb{D}} |I_\mu f|^2 \, d\mu \leq \|\mathcal{T}_\mu\| \|f\|_2^2.$$

Suppose that (f_n) is a sequence in $\mathcal{D}(\mathcal{T}_\mu)$ which converges to f in H^2. Then $f_n(z) \to f(z)$ for every $z \in \mathbb{D}$. On the other hand, since I_μ is bounded, we have $I_\mu f_n (= f_n) \to I_\mu f$ in $L^2(\mathbb{D}, \mu)$. It follows from Fatou's lemma that

$$\int_{\mathbb{D}} |I_\mu f - f|^2 \, d\mu \leq \liminf_{n\to\infty} \int_{\mathbb{D}} |I_\mu f - f_n|^2 \, d\mu = \liminf_{n\to\infty} \left\| I_\mu f - f_n \right\|_{L^2(\mathbb{D},\mu)}^2 = 0.$$

Thus $I_\mu f = f$ μ-a.e. Hence we have $\int_{\mathbb{D}} |f|^2 \, d\mu \leq \|\mathcal{T}_\mu\| \|f\|_2^2$ for every $f \in H^2$, i.e., μ is a Carleson measure. $\square$

4.2 Hankel operators with symbols of measures

In this chapter, we introduce the definition of Hankel operators whose symbols are complex Borel measures on the unit disc. They have many properties similar to Toeplitz operators.

For $f \in A(\mathbb{D})$, define the function $H_\mu f$ on $\mathbb{D}$ by

$$H_\mu f(z) := \int_{\overline{\mathbb{D}}} \frac{f(w)}{1 - wz} \, d\mu(w) \qquad (z \in \mathbb{D}).$$

It follows that

$$
\begin{aligned}
H_\mu f(z) &= \int_{\overline{\mathbb{D}}} f(w) \sum_{n=0}^{\infty} w^n z^n \, d\mu(w) \\
&= \sum_{n=0}^{\infty} \int_{\overline{\mathbb{D}}} f(w) w^n \, d\mu(w) z^n = \sum_{n=0}^{\infty} (f \cdot \mu)_n z^n \qquad (z \in \mathbb{D}).
\end{aligned}
$$

Therefore $H_\mu f$ is analytic in $\mathbb{D}$. Now define

$$
\mathcal{D}(H_\mu) = \{f \in A(\mathbb{D}) : H_\mu f \in H^2\}.
$$

It is clear that $\mathcal{D}(H_\mu)$ is a linear subspace of H^2. The mapping H_μ is a linear operator on H^2 with domain $\mathcal{D}(H_\mu)$.

Definition 4.6. The linear operator H_μ is called the *Hankel operator with symbol* μ.

Lemma 4.7. *For every $\mu \in M(\overline{\mathbb{D}})$ and $f \in A(\mathbb{D})$,*

$$
H_\mu S f = S^* H_\mu f.
$$

Proof. For each $z \in \mathbb{D}$,

$$
\begin{aligned}
H_\mu f(z) - H_\mu f(0) &= \int_{\overline{\mathbb{D}}} \frac{f(w)}{1 - wz} \, d\mu(w) - \int_{\overline{\mathbb{D}}} f(w) \, d\mu(w) \\
&= z \int_{\mathbb{T}} \frac{(Sf)(w)}{1 - wz} \, d\mu(w) = z H_\mu S f(z),
\end{aligned}
$$

and hence

$$
H_\mu S f(z) = \frac{H_\mu f(z) - H_\mu f(0)}{z} = S^* H_\mu f(z). \qquad \square
$$

Proposition 4.8. *Let $\mu \in M(\overline{\mathbb{D}})$ and $\alpha \in \mathbb{C} \backslash \mathbb{T}$. Then the following hold:*

(a) *For $f \in A(\mathbb{D})$, $f \in \mathcal{D}(H_\mu)$ if and only if $(S - \alpha) f \in \mathcal{D}(H_\mu)$.*

(b) *For $f \in H^2$, $f \in \mathrm{cl}_{H^2}(\mathcal{D}(H_\mu))$ if and only if $(S - \alpha) f \in \mathrm{cl}_{H^2}(\mathcal{D}(H_\mu))$.*

Proof. (a) Let $f \in A(\mathbb{D})$. Then, by Lemma 4.7,

$$\begin{aligned} H_\mu(S-\alpha)f &= H_\mu Sf - H_\mu \alpha f \\ &= S^* H_\mu f - \alpha H_\mu f = (S^* - \alpha) H_\mu f. \end{aligned}$$

Hence if $H_\mu f \in H^2$, then $H_\mu(S-\alpha)f \in H^2$. Conversely, suppose that $H_\mu(S-\alpha)f \in H^2$. Then the function

$$\begin{aligned} SH_\mu(S-\alpha)f &= S(S^*-\alpha)H_\mu f \\ &= H_\mu f - H_\mu f(0) - \alpha S H_\mu f \end{aligned}$$

also belongs to H^2. Hence we have $(1-\alpha S)H_\mu f \in H^2$. It follows that $H_\mu f \in H^2$. Therefore $f \in \mathcal{D}(H_\mu)$ if and only if $(S-\alpha)f \in \mathcal{D}(H_\mu)$.

(b) Same as the proof of Proposition 3.6(b). □

Remark 4.9.

(a) If we take $\alpha = 0$ in Proposition 4.8, it follows that the linear subspaces $\mathcal{D}(H_\mu)$ and $\mathrm{cl}_{H^2}(\mathcal{D}(H_\mu))$ are S-invariant.

(b) If $H_\mu f = 0$, then $H_\mu S f = 0$, i.e., the linear subspace

$$\ker H_\mu = \big\{ f \in \mathcal{D}(H_\mu) : H_\mu f = 0 \big\}$$

of H^2 is S-invariant.

Proposition 4.10. *Let $\mu \in M(\overline{\mathbb{D}})$. Then one of the following holds:*

(i) $\mathcal{D}(H_\mu) = \{0\}$.

(ii) $\mathcal{D}(H_\mu)$ *is dense in* H^2.

(iii) $\mathrm{cl}_{H^2}(\mathcal{D}(H_\mu)) = \theta H^2$, *where θ is a singular inner function.*

Proof. Same as the proof of Proposition 3.8. □

Proposition 4.11. *Suppose that $\mu \in M(\mathbb{T})$. If $\mathrm{cl}_{H^2}(\mathcal{D}(H_\mu))$ contains a polynomial, then $\mathcal{D}(H_\mu)$ is dense in H^2.*

Proof. Same as the proof of Proposition 3.9. □

Proposition 4.12. *Let $\mu \in M(\mathbb{D})$. Then*

$$\langle H_\mu f, g\rangle = \int_{\mathbb{D}} f(w)\overline{g(\overline{w})}\, d\mu(w)$$

for every $f \in \mathcal{D}(H_\mu)$ and $g \in A(\mathbb{D})$.

Proof. Same as the proof of Lemma 3.19. □

As a corollary, we have that if μ is nonnegative and $\operatorname{supp}\mu \subseteq (-1,1)$, then H_μ is nonnegative.

Chapter 5
Concluding remarks and open questions

In this chapter, we provide some concluding remarks and open questions.

5.1 The triviality of the discrete symbol cases

In view of Proposition 3.16, the following conjecture arises naturally: *Every Toeplitz operator with discrete symbol is trivial.*

We already know that if $m(\operatorname{supp}\mu) = 0$, then T_μ is trivial. Let $\mu \in M(\mathbb{T})$ be a discrete measure whose support has only finitely many limit points, for example,

$$\mu = \sum_{n=1}^{\infty} \frac{1}{2^n} \delta_{\zeta_n},$$

where $\zeta_n = e^{\pi i/2^n}$. By an argument similar to the proof of Proposition 3.16, we can show that

$$\mathcal{D}(T_\mu) = \big\{ f \in A(\mathbb{D}) : f(\zeta) = 0 \text{ for all } \zeta \in \operatorname{supp}\mu \big\},$$

and $T_\mu f = 0$ for all $f \in \mathcal{D}(T_\mu)$. Hence T_μ is trivial.

Note that every polynomial has only finitely many zeros. It follows that $\mathcal{D}(T_\mu)$ cannot contain any polynomial. Nevertheless, $\mathcal{D}(T_\mu)$ contains a nonzero function by Fatou's theorem for $A(\mathbb{D})$, which says that, for any given closed set $K \subseteq \mathbb{T}$ with $m(K) = 0$, there exists a function in $A(\mathbb{D})$ which vanishes precisely on K (cf. [Hof]). But it does not seem to be easy to determine whether $\mathcal{D}(T_\mu)$ is dense in H^2 or not. We thus may ask:

Question 5.1. Is the domain

$$\mathcal{D}(T_\mu) = \left\{ f \in A(\mathbb{D}) : f(e^{\pi i/2^n}) = 0 \text{ for all } n \in \mathbb{N} \right\}$$

dense in H^2?

5.2 Truncated Toeplitz operators

We here give two questions on the truncated Toeplitz operators with symbols of measures.

First of all, related to the equivalence (3.5.1) we may ask:

Question 5.2. Is (3.5.1) still true for the general case (i.e., $\dim \mathcal{H}(\theta) = \infty$) ?

For this question, we are tempted to guess:

Conjecture 5.3. *Let θ be a nonconstant inner function and let $\mu \in M(\mathbb{T})$. Then*

$$T_\mu^\theta \textit{ is a bounded operator on } \mathcal{H}(\theta) \iff \mu \textit{ is a Carleson measure for } \mathcal{H}(\theta).$$

In this case, $T_\mu^\theta = A_\mu$ on $\mathcal{D}(T_\mu^\theta)$.

The following question also seems to be not easy to answer:

Question 5.4. What condition on the symbol φ quarantees the boundedness of a truncated Toeplitz operator A_φ?

Also we would like to conjecture:

Conjecture 5.5. *A_φ is bounded if and only if the measure $\varphi{\cdot}m$ is a Carleson measure for $\mathcal{H}(\theta)$.*

5.3 Symmetric closed operators which are not self-adjoint

One of the lonstanding problems in the unbounded operator theory is to find fruitful examples of a symmetric and closed operator which is not self-adjoint. There are few examples of a symmetric closed operator which is not self-adjoint (see [Con2, Example X.1.11, p306]). In this viewpoint we may ask:

Question 5.6. Does there exist a Toeplitz operator with symbol of measure that is a symmetric and closed operator which is not self-adjoint ?

However we were unable to answer the above question. We note that T_μ is symmetric whenever μ is a real measure. But since the domain of T_μ is the disc algbra, it seems to be difficult to find a measure μ for which T_μ is a closed operator. Thus we need to consider the closure of the operator T_μ. But, unsatisfactorily, the closure of T_μ is liable to be self-adjoint. For example, if $\mu = \delta_1$, then $T_\mu = 0|_{\mathcal{D}(T_\mu)}$, where $\mathcal{D}(T_\mu) = \{f \in A(\mathbb{D}) : f(1) = 0\}$. Thus, T_μ is not closed and $\operatorname{cl} T_\mu = 0$, and hence T_μ is self-adjoint. If instead $\mu = m$, then $T_m = I|_{A(\mathbb{D})}$. Thus also T_μ is not closed, but $\operatorname{cl} T_\mu = I_{H^2}$, which is self-adjoint. What about the case $\mu = \varphi \cdot m$, where $\varphi = (1-z)^{-1/2} \in H^1 \backslash H^2$. Then we have $\mathcal{D}(T_\mu) = \{f \in A(\mathbb{D}) : \varphi f \in H^2\}$ and $\mathcal{D}(T_\varphi) = \{f \in H^2 : \varphi f \in H^2\}$. Thus we can easily check that T_φ is closed. But we were not able to decide whether T_μ is closed.

We may also ask:

Question 5.7. Does there exist a symmetric Toeplitz operator with symbol of measure whose closure is not self-adjoint ?

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국문초록

이 학위 논문에서 다Ł는 내용은 심벌이 보 측도인 토에플¬츠와 한켈 작용소의 정의와 그들이 갖는 여 가지 성질들이다. 단위원 상의 측도 μ 가 주어져 있을 때, 이| 심벌\ 갖는 토에플¬츠 작용소 T_μ | 하디 공간 H^2 상의 (비유계) 선형 작용소\서 정의한다. 이 때, T_μ 는 조밀하게 정의되지 않을지도 모x다. 그 지Ì, T_μ 의 정의역이 항상 특별한 형태| 갖는다는 것을 확인하다. 이 논문의 중심적인 질문은 선형 작용소\ 정의한 토에플¬츠 작용소 T_μ 가 언제 그 정의역에서 유계인지 묻는 것이다. 이 질문에 대한 답은 측도의 호환 가능성과 관(이 있다는 것을 보인다: T_μ 의 정의역이 다항식을 포함하는 경우, T_μ 가 그 정의역에서 유계일 필요충분조건은 심벌 μ 가 단위원 상의 칼 슨 측도인 것이다. 다x 하나의 질문은 측도| 심벌\ 갖는 토에플¬츠 작용소는 함수| 심벌\ 갖는 토에플¬츠 작용소와 얼È나 다x지 묻는 것이다. 이 질문에 대한 답으\, 단위원 상의 측도 μ 가 t벡 측도에 대하여 특이 측도이면, Î은 경우 작용소 T_μ 는 자명한 작용소인 것을 보인다. 또한, 단위원 상의 보 측도| 심벌\ 갖는 토에플¬츠 작용소들과 삼각 모멘트 문제의 관(성을 연구한다. 한켈 작용소에 대해서도 대응되는 정의| 제시하고 여 가지 성질들을 확인한다.

주요어: 토에플¬츠 작용소, 한켈 작용소, 하디 공간, 재생핵, 비유계 작용소, 보 측도, 칼 슨 측도, 모멘트 문제.

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